J. Math. Anal. Appl. 429 (2015) 366–382

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# The universal back-projection formula for spherical means and the wave equation on certain quadric hypersurfaces

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A R T I C L E   I N F O

*Article history:*
Received 2 November 2014
Available online 9 April 2015
Submitted by B. Kaltenbacher

*Keywords:*
Spherical means
Wave equation
Radon transform
Computed tomography
Inversion formula
Universal backprojection

A B S T R A C T

In many tomographic applications and elsewhere, there arises a need to reconstruct a function from the data on the boundary of some domain given either by the spherical means of the function, or by the corresponding solution of the free-space wave equation. In this paper, we show that the so-called universal back-projection formulas provide exact recovery of the unknown function with compact support for data on any quadric hypersurfaces that can be approximated by elliptic hypersurfaces. These quadric hypersurfaces include elliptic paraboloid as well as parabolic and elliptic cylinders.



## 1. Introduction

Consider a convex domain $\Omega \subset \mathbb{R}^d$, where $d \geq 2$ denotes the spatial dimension, with smooth boundary $\partial\Omega$. Let $C_c^\infty(\Omega)$ denote the set of all real valued smooth functions $f: \mathbb{R}^d \to \mathbb{R}$ that are compactly supported in $\Omega$. In the present paper, we deal with the problem of reconstructing an unknown function $f \in C_c^\infty(\Omega)$ from the data on the boundary $\partial\Omega$, which either consists of the spherical means of $f$ or the solution of the standard free-space wave equation with initial data $(f, 0)$. In particular, we investigate the universal back-projection formula (see [20,21,27,31,33,46]) on quadric hypersurfaces that can be approximated by elliptic hypersurfaces. For such type of quadrics we will show that the universal back-projection formula provides an exact reconstruction.

### 1.1. Inversion from spherical means and the wave equation

We consider the spherical means operator $\mathcal{M}: C_c^\infty(\Omega) \to C^\infty(\mathbb{R}^n \times (0, \infty))$ that maps a function $f \in C_c^\infty(\Omega)$ to the spherical means $\mathcal{M}f: \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ defined by

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http://dx.doi.org/10.1016/j.jmaa.2015.04.018

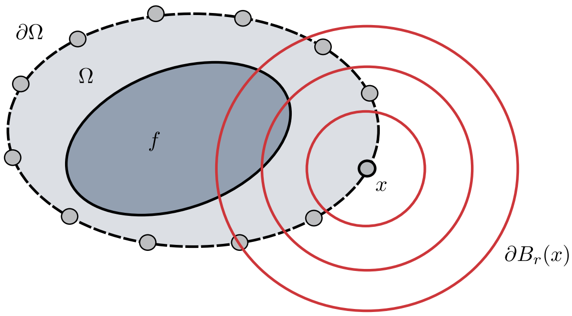


**Fig. 1.** *Illustration of the reconstruction problems.* The function $f$ is supported inside the domain $\Omega$. Detectors are placed on the boundary $\partial\Omega$ of the domain and record either averages of $f$ over spherical surfaces centered at $\partial\Omega$ or the solution of the wave equation restricted to $\partial\Omega \times (0, \infty)$.

$$(\mathcal{M} f)(x, r) := \frac{1}{\omega_{d-1}} \int_{S^{d-1}} f(x + ry)\, \mathrm{d}s(y), \qquad \text{for } (x, r) \in \mathbb{R}^d \times (0, \infty)\,.$$

Here $S^{d-1} \subset \mathbb{R}^d$ is the $(d-1)$-dimensional unit sphere, $\omega_{d-1}$ is its total surface area, and $\mathrm{d}s$ denotes the standard surface measure. We also consider the solution operator $\mathcal{W}: f \mapsto p$ of the standard free-space wave equation, that maps a function $f \in C_c^\infty(\Omega)$ to the solution $p: \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ of the following initial value problem

$$\begin{cases} (\partial_t^2 - \Delta_x)\, p(x, t) = 0 & \text{for } (x, t) \in \mathbb{R}^d \times (0, \infty), \\ p(x, 0) = f(x) & \text{for } x \in \mathbb{R}^d, \\ (\partial_t p)(x, 0) = 0 & \text{for } x \in \mathbb{R}^d\,. \end{cases} \tag{1.1}$$

With the operators just introduced, the considered reconstruction problems can be formulated as follows: recover the unknown function $f \in C_c^\infty(\Omega)$ from either its restricted spherical means

$$m(x, r) = (\mathcal{M} f)(x, r) \qquad \text{for } (x, r) \in \partial\Omega \times (0, \infty)\,, \tag{1.2}$$

or the corresponding wave data

$$p(x, t) = (\mathcal{W} f)(x, t) \qquad \text{for } (x, t) \in \partial\Omega \times (0, \infty)\,. \tag{1.3}$$

These two problems are essentially equivalent because the solution operator $\mathcal{W}$ can be expressed through the spherical means operator $\mathcal{M}$, and vice versa (Fig. 1).

The problems of recovering a function from the spherical means (1.2) or the wave data (1.3) arise, for example, in the so-called photoacoustic tomography (PAT) and thermoacoustic tomography (TAT), where the unknown function $f$ represents the initial pressure of an ultrasonic wave that is induced by a short electromagnetic pulse. In PAT/TAT using point-like detectors these problem arise in three spatial dimensions (see, for example, [15,26,47]). When using linear of circular integrating detectors in PAT/TAT, these reconstruction problems arise in two spatial dimensions (see [11,19,40,49]). The problems of recovering a function from its spherical means of the wave data also arises in other technologies, such as SONAR [8,42], SAR imaging [5,43,45], ultrasound tomography [35,36], and seismic imaging [9,13].

The derivation of explicit inversion formulas for recovering a function from data (1.2) or (1.3) has recently been addressed by many authors. Such formulas are currently only known for special boundaries $\partial\Omega$. For example, explicit inversion formulas have been derived for hyperplanes [5,7,10,13,30,46], spheres [16,17, 27,32,46] and cylinders [36]. Reconstruction formulas for some polygons and polyhedra in two and three spatial dimensions have been obtained in [29]. Recently, explicit formulas for inverting (1.2) and (1.3)