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The consistency of the nearest neighbor estimator of the density function based on WOD samples ☆

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ABSTRACT

In this paper, the consistency of the nearest neighbor estimator of the density function based on widely orthant dependent (WOD, in short) samples is investigated. The convergence rate of strong consistency, the complete consistency, the uniformly complete consistency and uniformly strong consistency of the nearest neighbor estimator of the density function based on WOD samples are established. Our results established in the paper generalize or improve the corresponding ones for independent samples and some negatively dependent samples.

1. Introduction

1.1. Brief review

Suppose that the population X has the unknown density function $f(x)$ and $X_1, X_2, \cdots, X_n$ are samples of X. Let $\{k_n, n \geq 1\}$ be a sequence of positive integers such that $1 \leq k_n \leq n$. For fixed x and n, denote

$$a_n(x) = \min\{a : \text{there exist at least } k_n' i \text{ such that } X_i \in [x-a, x+a]\}.$$

Loftsgarden and Quesenberry [14] introduced the nearest neighbor estimator of $f(x)$ as follows:

$$f_n(x) = \frac{k_n}{2na_n(x)}. \tag{1.1}$$

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Since the concept of the nearest neighbor estimator of the density function was introduced by Loftsgarden and Quesenberry [14], many authors devoted to study the asymptotic properties of the nearest neighbor estimator of the density function. For independent samples, Loftsgarden and Quesenberry [14] and Wagner [19] established the weak consistency and strong consistency, respectively; Moore and Henrichon [15] and Devroye and Wagner [6] obtained the uniform consistency and strong uniform consistency, respectively; Chen [3] established the convergence rate of the consistency of the nearest neighbor density estimator, and so on. For dependent samples, Boente and Fraiman [1] studied the strong consistency of the nearest neighbor density estimator based on φ-mixing and α-mixing samples; Chai [2] obtained the strong consistency, weak consistency, uniformly strong consistency and convergence rate of the nearest neighbor density estimator based on φ-mixing stationary processes; Yang [26] investigated the weak consistency, strong consistency, uniformly strong consistency of the nearest neighbor density estimator based on negatively associated (NA, in short) samples; Liu and Zhang [13] established the consistency and asymptotic normality of nearest neighbor density estimator based on φ-mixing samples, and so on.

Recently, Yang [26] established the following weak consistency and strong consistency for NA samples.

Theorem A. *Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables and $k_n \to \infty$, $\frac{k_n}{n} \to 0$ as $n \to \infty$.*

(i) If $\frac{k_n^2}{n} \to \infty$ as $n \to \infty$, then

$$f_n(x) \xrightarrow{P} f(x). \tag{1.2}$$

(ii) If for any $\gamma > 0$,

$$\sum_{n=1}^{\infty} \exp\left\{-\frac{\gamma k_n^2}{n}\right\} < \infty, \tag{1.3}$$

then

$$f_n(x) \to f(x) \quad a.s., \text{ as } \quad n \to \infty. \tag{1.4}$$

The main purpose of this article is to generalize and improve the result of Theorem A for NA random variables to the case of widely orthant dependent (WOD) random variables, which includes NA as a special case. In addition, we will present the convergence rate of strong consistency, uniformly complete consistency and uniformly strong consistency for the nearest neighbor estimator of density function based on WOD samples. Our results generalize or improve the corresponding ones of Yang [26] for NA random variables to the case of WOD random variables. The key techniques used in the paper are the Bernstein type inequality and the truncated method.

1.2. Concepts of widely orthant dependence

In this subsection, we will recall the definition of WOD random variables, which was introduced by Wang et al. [23] as follows.

Definition 1.1. For the random variables $\{X_n, n \geq 1\}$, if there exists a finite real sequence $\{g_U(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$P(X_1 > x_1, X_2 > x_2, \cdots, X_n > x_n) \leq g_U(n) \prod_{i=1}^{n} P(X_i > x_i),$$

then we say that the $\{X_n, n \geq 1\}$ are widely upper orthant dependent (WUOD, in short); if there exists a finite real sequence $\{g_L(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,