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Upper bounds for the distances between consecutive zeros of solutions of first order delay differential equations [☆]

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ABSTRACT

By introducing a new class of sequences involving iterates of the delay, we are able to derive a sharper estimate on the ratio $x(\tau(t))/x(t)$. Based on this, we get some new upper bounds for the distance between consecutive zeros of solutions of the first order delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0.$$

In particular, our results are not covered by previously known results. Some examples and a table are given to illustrate our results.

1. Introduction

In this paper, we will investigate the upper bound for the distance between consecutive zeros of solutions of a first order differential equation with a variable delay of the form

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \ge t_0, \tag{1.1}$$

where $p(t),\ \tau(t) \in C([t_0,\ \infty),\ [0,\ \infty))$, $\tau(t) < t$, $\tau(t)$ is strictly increasing and $\lim_{t\to\infty} \tau(t) = \infty$.

Throughout the remainder of the paper we will need the iterates of the inverses of the functions $\tau(t)$. For simplicity of notation in the lemmas, theorems, and examples that follow, we use the notation $\tau^0(t) = t$ and inductively define the iterates $\tau^{-i}(t)$:

$$\tau^{-i}(t) = \tau^{-1}\left(\tau^{-(i-1)}(t)\right), \quad i = 1, 2, \cdots.$$

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In general, let $C = C([\tau(t_0),\ t_0],\ R)$ denote the Banach space consisting of all continuous functions from $[\tau(t_0),\ t_0]$ to R, which is also called the Phase Space of Eq. (1.1). Using the method of steps, for the case of a constant delay of the form $\tau(t) = t - \tau$ and $\tau > 0$ is constant, it follows that for every continuous function $\varphi \in C([\tau(t_0),\ t_0],\ R)$, there exists a unique solution of Eq. (1.1) valid for $t \geq t_0$. For further questions on existence, uniqueness, and continuous dependence, see J.K. Hale [5] and L.H. Erbe [4].

We recall that a solution of Eq. (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise, it is called nonoscillatory. Eq. (1.1) is said to be oscillatory if every function that satisfies the equation eventually, that is, on the interval $[t_0, \infty)$, has arbitrarily large zeros. It is not sufficient only to know that some solutions have this behavior. It is well known that Eq. (1.1) is oscillatory provided one of the following conditions (see e.g. [1,2,6]) holds:

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e} \quad \text{or} \quad \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1.$$

There is an obvious gap between the two conditions when the limit $\lim\limits_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds$ does not exist. It is therefore interesting to consider the behavior of solutions of Eq. (1.1) under the following conditions (see e.g. [1,2,6,11]):

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \leq \frac{1}{e} \quad \text{and} \quad \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \leq 1. \tag{1.2}$$

We also obtain some results in these cases.

The purpose of this paper is to obtain a better upper bound estimate for the interval-length of successive zeros of the solutions of Eq. (1.1). Recently, there has been an increasing interest in studying the distribution of zeros of solutions of first order delay or neutral differential equations. Instead of using the following condition (see e.g. [2,3,6–11]) for Eq. (1.1):

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \geq \rho > 0, \quad t \geq t_0, \tag{1.3}$$

in what follows, we shall assume there exists a sequence of nonnegative numbers $\rho_i \geq 0$ for $i = 1, 2, \cdots$, such that

$$\int_{t}^{\tau^{-1}(t)} p(s_1) \int_{\tau(s_1)}^{t} p(s_2) \cdots \int_{\tau(s_{i-1})}^{\tau^{i-2}(t)} p(s_i)ds_i \cdots ds_2 ds_1 \geq \rho_i, \quad t \geq t_0. \tag{1.4}$$

If the condition (1.3) holds, then we can take $\rho_1 = \rho$ and $\rho_2 = \dfrac{\rho^2}{2!}$ (see e.g. [10,11]). In particular, for the following equation with constant coefficient and constant delay

$$x'(t) + px(t - \tau) = 0, \quad t \geq t_0, \tag{1.5}$$

by a simple calculation, we have