J. Math. Anal. Appl. 429 (2015) 576–590

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The order of magnitude for moments for certain cotangent sums

Helmut Maier [a], Michael Th. Rassias [b,c,*]

[a] Department of Mathematics, University of Ulm, Helmholtzstrasse 18, 89081 Ulm, Germany
[b] Department of Mathematics, ETH-Zürich, Rämistrasse 101, 8092 Zürich, Switzerland
[c] Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA

ARTICLE INFO

Article history:
Received 8 December 2014
Available online 15 April 2015
Submitted by B.C. Berndt

Keywords:
Cotangent sums
Equidistribution
Moments
Continued fractions
Measure

ABSTRACT

We settle a question on the rate of growth of the moments of cotangent sums considered by the authors in their previous papers [7,8].

1. Introduction

The authors in a joint work [7] and the second author in his thesis, investigated the distribution of cotangent sums

$$c_0\left(\frac{r}{b}\right) = -\sum_{m=1}^{b-1} \frac{m}{b} \cot\left(\frac{\pi m r}{b}\right),$$

as r ranges over the set

$$\{r \,:\, (r,b)=1, A_0 b \leq r \leq A_1 b\},$$

where A_0, A_1 are fixed with $1/2 < A_0 < A_1 < 1$ and b tends to infinity.

* Corresponding author.
E-mail addresses: helmut.maier@uni-ulm.de (H. Maier), michail.rassias@math.ethz.ch, michailrassias@math.princeton.edu (M.Th. Rassias).

http://dx.doi.org/10.1016/j.jmaa.2015.04.036
0022-247X/

Especially, they considered the moments

$$H_k = \lim_{b \to +\infty} \phi(b)^{-1} b^{-2k} (A_1 - A_0)^{-1} \sum_{\substack{A_0 b \le r \le A_1 b \\ (r,b)=1}} c_0 \left(\frac{r}{b}\right)^{2k}, \ k \in \mathbb{N},$$

where $\phi(\cdot)$ denotes the Euler phi-function.

They could show that all the moments H_k exist and that

$$\lim_{k \to +\infty} H_k^{1/k} = +\infty.$$

Thus the series $\sum_{k \ge 0} H_k x^{2k}$ converges only for $x = 0$.

It was left open, whether the series

$$\sum_{k \ge 0} \frac{H_k}{(2k)!} x^k \qquad (*)$$

converges for values of x different from 0 (cf. [8]). This fact would considerably simplify the proof for the distribution of the cotangent sums $c_0(r/b)$ (uniqueness of measures determined by their moments, see [4, Section 30, The Method of Moments, Theorem 30.1]).

Essential for the investigation was the result:

$$H_k = \int_0^1 \left(\frac{g(x)}{\pi}\right)^{2k} dx,$$

where

$$g(x) = \sum_{l \ge 1} \frac{1 - 2\{lx\}}{l}.$$

The function g has been also investigated in the paper [5] of R. de la Bretèche and G. Tenenbaum. More recently it has also been investigated by M. Balazard and B. Martin [1,2]. We are indebted to M. Balazard for this information. Their ideas, as well as ideas from the paper of S. Marmi, P. Moussa and J.-C. Yoccoz [9] will play an important role in our paper. Independently also S. Bettin [3] investigated this problem and obtained the positivity of the radius of convergence of the series (*). He could also replace the interval $(1/2, 1)$ for A_0, A_1 by the interval $(0, 1)$. We are thankful to S. Bettin for reading an earlier version of this paper and for providing useful remarks. We shall show the following theorem.

Theorem 1.1. *There are constants $c_1, c_2 > 0$, such that*

$$c_1 \Gamma(2k+1) \le \int_0^1 g(x)^{2k}\, dx \le c_2 \Gamma(2k+1),$$

for all $k \in \mathbb{N}$, where $\Gamma(\cdot)$ stands for the Gamma function.

Corollary 1.2. *The series*

$$\sum_{k \ge 0} \frac{H_k}{(2k)!} x^k$$

has radius of convergence π^2.