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Oscillation and non-oscillation of Euler type half-linear differential equations

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ABSTRACT

We investigate oscillatory properties of second order Euler type half-linear differential equations whose coefficients are given by periodic functions and functions having mean values. We prove the conditional oscillation of these equations. In addition, we prove that the known oscillation constants for the corresponding equations with only periodic coefficients do not change in the studied more general case. The presented results are new for linear equations as well.

1. Introduction

The Euler half-linear differential equation is an equation of the form

$$[\Phi(x')]' + \frac{\gamma}{t^p}\,\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn} x,\ p > 1. \tag{1.1}$$

This is one of a few half-linear differential equations which can be (at least partially) solved explicitly. A solution is considered in the form $x(t) = t^\lambda$. Substituting this function into Eq. (1.1), we obtain an algebraic equation for λ, where its solution gives the explicit value of λ. Another reason, why Eq. (1.1) appears so frequently in the half-linear oscillation theory, is that Eq. (1.1) is conditionally oscillatory. This means that there exists the so-called oscillation (or critical) constant $\Gamma \in \mathbb{R}$ with the property that Eq. (1.1) is oscillatory for $\gamma > \Gamma$ and non-oscillatory for $\gamma < \Gamma$. Note that, in the case of Eq. (1.1), the critical constant is

$$\Gamma = \gamma_p := \left(\frac{p-1}{p}\right)^p.$$

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The conditional oscillation implies that Eq. (1.1) is a good "comparison equation" in the oscillation theory of the general half-linear differential equation

$$[r(t)\Phi(x')]' + c(t)\Phi(x) = 0 \tag{1.2}$$

with continuous coefficients r, c and $r(t) > 0$ on an interval under consideration. As a special case of Eq. (1.2) for $p = 2$, we have the linear equation

$$[r(r)x']' + c(t)x = 0. \tag{1.3}$$

We remark that the classical Sturmian theory of Eq. (1.3) extends almost verbatim to Eq. (1.2) although the additivity of the solution space is lost (only the homogeneity remains). In fact, this property of the solution space is the reason why Eq. (1.2) is called *half-linear* (the name was introduced in [1]).

Oscillatory and non-oscillatory properties of various generalizations of the Euler equation (1.1) have been analyzed in numerous recent papers. We point out at least [5,9,10,20,22] and the references given therein. Explicitly, we recall the result of [11] which claims that the equation (sometimes called the generalized Riemann–Weber equation)

$$[\Phi(x')]' + \frac{1}{t^p}\left[\gamma_p + \sum_{i=1}^{n} \frac{\mu_i}{\operatorname{Log}^2_{(i)}(t)}\right]\Phi(x) = 0$$

is oscillatory if and only if there exists $k \in \{0, 1, \dots, n-1\}$ such that

$$\mu_i = \mu_p := \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} \quad \text{for } i \in \{1, \dots, k\}$$

and $\mu_{k+1} > \mu_p$. The reader can find the explanation of used notations in the next section (especially, see (2.1) below and Remark 2).

Motivated by the "linear case" (see [16]), where perturbations in the differential term are considered as well, the above result of [11] has been extended to the equation

$$\left[\left(1 + \sum_{i=1}^{n} \frac{\alpha_i}{\operatorname{Log}^2_{(i)}(t)}\right)\Phi(x')\right]' + \frac{1}{t^p}\left[\gamma_p + \sum_{i=1}^{n} \frac{\mu_i}{\operatorname{Log}^2_{(i)}(t)}\right]\Phi(x) = 0. \tag{1.4}$$

In [6], there is proved that Eq. (1.4) is oscillatory if and only if there exists $k \in \{0, 1, \dots, n-1\}$ for which

$$\mu_i - \gamma_p\alpha_i = \mu_p, \quad i \in \{1, \dots, k\}, \qquad \mu_{k+1} - \gamma_p\alpha_{k+1} > \mu_p.$$

A typical problem in the qualitative theory of various differential equations is what happens when constants (appearing as coefficients in an investigated equation) are replaced by periodic functions. This problem has been studied for Eq. (1.1) and Eq. (1.4) in [12] and [5], respectively. Roughly speaking, it has been shown that "nothing happens". More precisely, the results for equations with constant coefficients remain to hold when the constants are replaced by the mean values of periodic functions over a given period (i.e., all functions have the same period).

Instead of periodic perturbations, one can consider wider classes of perturbations (e.g., given by almost periodic and asymptotically almost periodic functions). This problem has been studied in the series of papers [14,15,20,22]. We continue in this research. In this paper, we show that, to obtain the oscillation constant for the perturbed Euler equations in the form of Eq. (1.4), it suffices to replace the last considered constants α_n, μ_n by the mean values of general continuous functions over $\mathbb{R}^+$.