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# Existence and uniqueness of solutions to stochastic Rayleigh–Plesset equations

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ABSTRACT

The goal of the present paper is to establish a mathematical basis for analyses of the Rayleigh–Plesset equation combined with certain types of stochastic terms for studying motions of a single bubble immersed in water. We show the unique existence of global solutions to the system and also the existence of invariant measures making use of a suitable Lyapunov function constructed for the underlying deterministic dynamics.



## 1. Introduction

In the present article, we are concerned with the Rayleigh–Plesset equation perturbed by random forces. The Rayleigh–Plesset equation [15,17] describes the radial motion of a single spherical bubble immersed in incompressible flows and is given by an ODE of

$$\frac{d^2R}{dt^2} + \frac{3}{2R}\left(\frac{dR}{dt}\right)^2 + \frac{4\nu_L}{R^2}\frac{dR}{dt} + \frac{2\sigma_W}{\rho_L R^2} - \frac{(p_V - p_\infty)}{\rho_L R} - \frac{p_{G_0}\bar{R}^{3k}}{\rho_L R^{3k+1}} = 0, \tag{1.1}$$

where the scalar-valued unknown function $R = R(t)$ represents a radius of the bubble at time $t$, and $\bar{R}$, $\nu_L$, $\sigma_W$, $\rho_L$, $p_V$, $p_\infty$, $p_{G_0}$ and $k$ are given constants whose meanings will be clarified in Section 2. This equation has long been extensively used in mechanical engineering and other industrial fields to analyze cavitation, one of the most important phenomena in bubbly flows. A number of analytical and numerical investigations

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of this equation have been done successfully to reproduce oscillations in the bubble radius [1,3,14,16]. The Rayleigh–Plesset equation has also been investigated from the point of view of dynamical systems [18].

Recent progress in engineering enables us to positively utilize very minute bubbles called microbubbles or nanobubbles for industrial and biomedical applications [20]. Since typical diameters of bubbles in these applications are less than several tens of micrometers, random molecular motions around each bubble and interactions among bubbles are considered to influence the bubble dynamics. The (re)discovery of sonoluminescence [8] around 1990 has also been a thrust in the study of bubble dynamics. Single-bubble sonoluminescence occurs by applying ultrasonic wave vibration externally to a bubble. Depending on relations between a typical bubble size and a magnitude of vibration, stable or chaotic behaviors of bubbles are observed. See a comprehensive review [4] for the details. To account for complex behaviors of bubbles, several models are incorporated into the Rayleigh–Plesset equation including thermal effects [2,21], spherical asymmetry [5,10] and phase transitions [4,11].

In view of these studies, we try to give a mathematical support for physical modeling and numerical studies by mathematically analyzing the original Rayleigh–Plesset equation but with a stochastic forcing as a generic model for more complex systems. Similar attempts are seen concerning the Duffing equation, the Liénard equation and the van del Pol equation in particular, appearing in mechanical and electrical engineering. Other applications include biology, population dynamics and economics to list just a few. Major difficulties of our problem come from the singularity of the coefficients in (1.1). In fact, to the best of the authors' knowledge concerning the Rayleigh–Plesset equation, fundamental mathematical properties as the existence and the uniqueness of global solutions and their behavior do not seem to be established even for the simplest equation (1.1).

The present paper is organized as follows. In Section 2, we explain how stochastic effects are taken into account following the standard argument of deriving the Rayleigh–Plesset equation. Then in Section 3, we show the global existence of the solution to (1.1) by constructing a suitable Lyapunov function and obtain asymptotic behaviors of a bubble radius subject to the magnitude of the liquid viscosity $\nu_L$. Based on the results for deterministic Rayleigh–Plesset equation, Section 4 is devoted to the investigation of several kinds of stochastic Rayleigh–Plesset equations derived in Section 2 and proves the unique existence of global solutions and the existence of an invariant measure. In Section 5, we obtain similar results for other possible forms of stochastic Rayleigh–Plesset equations than appeared in Section 4 expecting future developments.

## 2. Derivation of equations

In this section, we demonstrate how a noise can be introduced to the Rayleigh–Plesset equation. Among several possible random effects, we adopt a Brownian motion defined on a certain probability space.

### *2.1. Derivation of stochastic Rayleigh–Plesset equations*

We consider liquid flows around an oscillating spherical gas bubble whose radius is $R(t)$ at time $t$. The flows are assumed to be incompressible and spherically symmetric, and to occupy the domain which is specified by $r \in [R(t), \infty)$, where $r$ is the radial coordinate. Then the continuity and momentum equations reduce to

$$\frac{\partial}{\partial r}(r^2 u) = 0 \tag{2.1}$$

and

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho_L}\frac{\partial p}{\partial r} \tag{2.2}$$