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Non-autonomous maximal regularity for forms of bounded variation

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ABSTRACT

We consider a non-autonomous evolutionary problem

$$u'(t) + \mathcal{A}(t)u(t) = f(t), \qquad u(0) = u_0,$$

where V, H are Hilbert spaces such that V is continuously and densely embedded in H and the operator $\mathcal{A}(t)\colon V \to V'$ is associated with a coercive, bounded, symmetric form $\mathfrak{a}(t,.,.)\colon V \times V \to \mathbb{C}$ for all $t \in [0,T]$. Given $f \in L^2(0,T;H)$, $u_0 \in V$ there exists always a unique solution $u \in MR(V,V') := L^2(0,T;V) \cap H^1(0,T;V')$. The purpose of this article is to investigate whether $u \in H^1(0,T;H)$. This property of *maximal regularity in* H is not known in general. We give a positive answer if the form is of *bounded variation*; i.e., if there exists a bounded and non-decreasing function $g\colon [0,T] \to \mathbb{R}$ such that

$$\left|\mathfrak{a}(t,v,w) - \mathfrak{a}(s,v,w)\right| \le \left[g(t) - g(s)\right]\|v\|_V\|w\|_V \quad (0 \le s \le t \le T,\ v,w \in V).$$

In that case, we also show that $u(.)$ is continuous with values in V. Moreover we extend this result to certain perturbations of $\mathcal{A}(t)$.

1. Introduction

The aim of the present article is to study maximal regularity for evolution equations governed by non-autonomous forms. More precisely, let $T > 0$, let V, H be separable Hilbert spaces such that V is continuously and densely embedded in H and let

$$\mathfrak{a}\colon [0,T] \times V \times V \to \mathbb{C}$$

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be a *non-autonomous form*; i.e., $\mathfrak{a}(t,.,.)$ is sesquilinear for all $t \in [0,T]$ and $\mathfrak{a}(.,v,w)$ is measurable for all $v,w \in V$. Moreover we assume that there exist constants M and $\alpha > 0$ such that

$$\big|\mathfrak{a}(t,v,w)\big| \le M\|v\|_V\|w\|_V \quad \big(t \in [0,T],\ v,w \in V\big)$$

and

$$\operatorname{Re}\mathfrak{a}(t,v,v) \ge \alpha\|v\|_V^2 \quad \big(t \in [0,T],\ v \in V\big).$$

Then for $t \in [0,T]$ we may define the *associated operator* $\mathcal{A}(t) \in \mathcal{L}(V,V')$ of $\mathfrak{a}(t,.,.)$ by

$$\big\langle \mathcal{A}(t)v,w\big\rangle = \mathfrak{a}(t,v,w) \quad (v,w \in V).$$

Here V' denotes the antidual of V and $\langle .,.\rangle$ denotes the duality between V' and V. Note that $H^1(0,T;V') \hookrightarrow C([0,T];V')$, so we may identify every element of $H^1(0,T;V')$ by its continuous representative. Now a classical result of Lions (see [8, p. 513], [17, p. 112]) states the following.

Theorem 1.1. *For every $f \in L^2(0,T;V')$ and $u_0 \in H$ there exists a unique*

$$u \in MR\big(V,V'\big) := L^2(0,T;V) \cap H^1\big(0,T;V'\big)$$

such that

$$\begin{cases} u' + \mathcal{A}u = f \quad \text{in } L^2\big(0,T;V'\big) \\ u(0) = u_0. \end{cases} \tag{1.1}$$

Moreover $MR(V,V') \hookrightarrow C([0,T];H)$ and

$$\|u\|^2_{L^2(0,T;V)} \le \frac{1}{\alpha^2}\|f\|_{L^2(0,T;V')} + \frac{1}{\alpha}\|u_0\|^2_H. \tag{1.2}$$

Let $f \in L^2(0,T;H)$, $u_0 = 0$ and let $u \in MR(V,V')$ be the solution of (1.1). In the autonomous case; i.e., if $\mathfrak{a}(t,.,.) = \mathfrak{a}(0,.,.)$ for all $t \in [0,T]$, it is well known that u is already in $H^1(0,T;H)$. Thus the question arises whether u is in $H^1(0,T;H)$ also in the non-autonomous case. This question seems still to be open and was explicitly asked by Lions [14, p. 68] in the case that $\mathfrak{a}(t,.,.)$ is symmetric for all $t \in [0,T]$. We say that $\mathfrak{a}$ has *maximal regularity* in H if for all $f \in L^2(0,T;H)$ and $u_0 = 0$ the solution u of (1.1) is in $H^1(0,T;H)$, and consequently in

$$MR_{\mathfrak{a}}(H) := \big\{u \in L^2(0,T;V) \cap H^1(0,T;H) : \mathcal{A}u \in L^2(0,T;H)\big\}.$$

It is easy to see that $\mathfrak{a}$ has maximal regularity in H implies that the solution u of (1.1) is in $H^1(0,T;H)$ for every $f \in L^2(0,T;H)$ and $u_0 \in Tr_{\mathfrak{a}}$, where $Tr_{\mathfrak{a}} := \{v(0) : v \in MR_{\mathfrak{a}}(H)\}$.

In the present article the contribution to this question is the following. Assume additionally that $\mathfrak{a}(t,.,.)$ is symmetric for all $t \in [0,T]$ and of *bounded variation*; i.e., there exists a bounded and non-decreasing function $g: [0,T] \to \mathbb{R}$ such that

$$\big|\mathfrak{a}(t,v,w) - \mathfrak{a}(s,v,w)\big| \le \big[g(t) - g(s)\big]\|v\|_V\|w\|_V \quad (0 \le s \le t \le T,\ v,w \in V).$$

Then $\mathfrak{a}$ has maximal regularity in H and $Tr_{\mathfrak{a}} = V$. Moreover $MR_{\mathfrak{a}}(H)$ is continuously embedded in $C([0,T];V)$ (see Theorem 4.1). The fact that the solution is continuous with values in V is not obvious at all and plays a central role in the following results. In Theorem 5.1 we extend this regularity result to