J. Math. Anal. Appl. 425 (2015) 55–70

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Compositions and convex combinations of averaged nonexpansive operators

Patrick L. Combettes [a,*], Isao Yamada [b]

[a] Sorbonne Universités – UPMC Univ. Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005 Paris, France

[b] Tokyo Institute of Technology, Department of Communications and Computer Engineering, Tokyo 152-8550, Japan

ARTICLE INFO

Article history:
Received 18 July 2014
Available online 22 November 2014
Submitted by H. Frankowska

Keywords:
Averaged operator
Fixed-point algorithm
Forward–backward splitting
Monotone operator
Nonexpansive operator

ABSTRACT

Properties of compositions and convex combinations of averaged nonexpansive operators are investigated and applied to the design of new fixed point algorithms in Hilbert spaces. An extended version of the forward–backward splitting algorithm for finding a zero of the sum of two monotone operators is obtained.

1. Introduction

Since their introduction in [3], averaged nonexpansive operators have proved to be very useful in the analysis and the numerical solution of problems arising in nonlinear analysis and its applications; see, e.g., [2,4–8,11,14–16,18–21].

Definition 1.1. Let $\mathcal{H}$ be a real Hilbert space, let D be a nonempty subset of $\mathcal{H}$, let $\alpha \in \,]0,1[$, and let $T : D \to \mathcal{H}$ be a nonexpansive (i.e., 1-Lipschitz) operator. Then T is averaged with constant α, or α-averaged, if there exists a nonexpansive operator $R : D \to \mathcal{H}$ such that $T = (1-\alpha)\,\mathrm{Id} + \alpha R$.

As discussed in [6,11,16], averaged operators are stable under compositions and convex combinations and such operations form basic building blocks in various composite fixed point algorithms. The averagedness constants resulting from such operations determine the range of the step sizes and other parameters in such algorithms. It is therefore important that they be tight since these parameters have a significant impact on the speed of convergence.

* Corresponding author. Fax: +33 1 4427 7200.
E-mail addresses: plc@ljll.math.upmc.fr (P.L. Combettes), isao@sp.ce.titech.ac.jp (I. Yamada).

http://dx.doi.org/10.1016/j.jmaa.2014.11.044
0022-247X/

In this paper, we discuss averagedness constants for compositions and convex combinations of averaged operators and construct novel fixed point algorithms based on these constants. In particular, we obtain a new version of the forward–backward algorithm with an extended relaxation range and iteration-dependent step sizes.

Throughout the paper, $\mathcal{H}$ is a real Hilbert space with scalar product $\langle \cdot \mid \cdot \rangle$ and associated norm $\|\cdot\|$. We denote by Id the identity operator on $\mathcal{H}$ and by d_S the distance function to a set $S \subset \mathcal{H}$; $\rightharpoonup$ and $\to$ denote, respectively, weak and strong convergence in $\mathcal{H}$.

2. Compositions and convex combinations of averaged operators

We first recall some characterizations of averaged operators (see [11, Lemma 2.1] or [6, Proposition 4.25]).

Proposition 2.1. *Let D be a nonempty subset of $\mathcal{H}$, let $T : D \to \mathcal{H}$ be nonexpansive, and let $\alpha \in \left]0,1\right[$. Then the following are equivalent:*

(i) *T is α-averaged.*
(ii) *$(1 - 1/\alpha)\operatorname{Id} + (1/\alpha)T$ is nonexpansive.*
(iii) $(\forall x \in D)(\forall y \in D)\ \|Tx - Ty\|^2 \leqslant \|x - y\|^2 - \frac{1-\alpha}{\alpha}\|(\operatorname{Id} - T)x - (\operatorname{Id} - T)y\|^2$.
(iv) $(\forall x \in D)(\forall y \in D)\ \|Tx - Ty\|^2 + (1 - 2\alpha)\|x - y\|^2 \leqslant 2(1 - \alpha)\langle x - y \mid Tx - Ty\rangle$.

The next result concerns the averagedness of a convex combination of averaged operators.

Proposition 2.2. *Let D be a nonempty subset of $\mathcal{H}$, let $(T_i)_{i\in I}$ be a finite family of nonexpansive operators from D to $\mathcal{H}$, let $(\alpha_i)_{i\in I}$ be a family in $\left]0,1\right[$, and let $(\omega_i)_{i\in I}$ be a family in $\left]0,1\right]$ such that $\sum_{i\in I}\omega_i = 1$. Suppose that, for every $i \in I$, T_i is α_i-averaged, and set $T = \sum_{i\in I}\omega_i T_i$ and $\alpha = \sum_{i\in I}\omega_i\alpha_i$. Then T is α-averaged.*

Proof. For every $i \in I$, there exists a nonexpansive operator $R_i : D \to \mathcal{H}$ such that $T_i = (1 - \alpha_i)\operatorname{Id} + \alpha_i R_i$. Now set $R = (1/\alpha)\sum_{i\in I}\omega_i\alpha_i R_i$. Then R is nonexpansive and

$$\sum_{i\in I}\omega_i T_i = \sum_{i\in I}\omega_i(1 - \alpha_i)\operatorname{Id} + \sum_{i\in I}\omega_i\alpha_i R_i = (1 - \alpha)\operatorname{Id} + \alpha R. \tag{2.1}$$

We conclude that T is α-averaged. □

Remark 2.3. In view of [8, Corollary 2.2.17], Proposition 2.2 is equivalent to [8, Theorem 2.2.35], and it improves the averagedness constant of [11, Lemma 2.2(ii)] which was $\alpha = \max_{i\in I}\alpha_i$. In the case of two operators, Proposition 2.2 can be found in [16, Theorem 3(a)].

Next, we turn our attention to compositions of averaged operators, starting with the following result, which was obtained in [16, Theorem 3(b)] with a different proof.

Proposition 2.4. *Let D be a nonempty subset of $\mathcal{H}$, let $(\alpha_1, \alpha_2) \in \left]0,1\right[^2$, let $T_1 : D \to D$ be α_1-averaged, and let $T_2 : D \to D$ be α_2-averaged. Set*

$$T = T_1T_2 \quad \textit{and} \quad \alpha = \frac{\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2}{1 - \alpha_1\alpha_2}. \tag{2.2}$$

Then $\alpha \in \left]0,1\right[$ and T is α-averaged.