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Elliptic PDEs with constant coefficients on convex polyhedra via the unified method

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ABSTRACT

We provide a new method to study the classical Dirichlet problem for constant coefficient second order elliptic PDEs on convex polyhedrons. Our approach is heavily motivated by Fokas' unified method for boundary value problems, and can be interpreted as the Fourier analogue to the classical boundary integral equations. The central object in this approach is the global relation: an integral equation which couples the known boundary data and the unknown boundary values. This integral equation depends holomorphically on two complex parameters, and the resulting analysis takes place on a Banach space of complex analytic functions closely related to the classical Paley–Wiener space. We write the global relation in the form of an operator equation and show that the analysis can be reduced to the case of Laplace's equation, from which the more general problem turns out to be a compact perturbation. We give a new integral representation to the solution to the underlying boundary value problem which serves as a concrete realisation of the fundamental principle of Ehrenpreis for all constant coefficient elliptic PDEs on convex polyhedra.

1. Introduction

Let $\Omega \subset \mathbf{R}^3$ be a convex polyhedron with faces $\{\Sigma_i\}_{i=1}^n$ and let $P(D)$ be a second order constant coefficient, elliptic differential operator. Given $f_i \in H^1(\Sigma_i)$ for $i = 1, \ldots, n$ such that $f_i = f_j$ on $\overline{\Sigma}_i \cap \overline{\Sigma}_j$ we consider the boundary value problem

$$P(D)u = 0, \quad \text{in } \Omega, \tag{1a}$$

$$u = f_i, \quad \text{on } \Sigma_i \text{ for } i = 1, \ldots, n. \tag{1b}$$

Here we use the standard notation $D = (-\mathrm{i}\partial_1, -\mathrm{i}\partial_2, -\mathrm{i}\partial_3)^t$. This boundary value problem gives rise to the so-called Dirichlet–Neumann map. This linear map is realised by the Steklov–Poincaré operator S, which

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transforms the *known* Dirichlet data into the *unknown* Neumann data [13,16]

$$S : H^1(\partial\Omega) \to L^2(\partial\Omega) : u|_{\partial\Omega} \mapsto \partial u/\partial\mathbf{n}|_{\partial\Omega}.$$

We will analyse this map within the framework of Fokas' unified method for elliptic boundary value problems [8,10], building on the results in [2,3]. We refer to [6] for a more traditional treatment of elliptic boundary value problems on polyhedra. Within Fokas' unified approach is the *global relation*: an integral equation defined on the set

$$Z_P = \{\boldsymbol{\lambda} \in \mathbf{C}^3 : P(\boldsymbol{\lambda}) = 0\}$$

that couples the unknown boundary values $\partial u/\partial\mathbf{n}_i|_{\Sigma_i}$ and the known boundary data f_i. In Section 2 we construct the global relation for the boundary value problem (1) and write it in terms of an abstract operator equation $T\Phi = \Psi$, where the vector Φ is related to the unknown Neumann boundary values and the vector Ψ is a known function of the Dirichlet boundary data. Functional analytic properties of the operator $T : X \to Y$, for appropriate Banach spaces X, Y, are obtained in Section 3 and it is these results that allow us to prove well-posedness of the underlying boundary value problem. In Section 4 we offer a practical Galerkin method based on this approach that can be used to obtain approximate solutions.

In addition to studying the Dirichlet–Neumann map via the global relation, we also provide new integral representations for the solutions to the boundary value problem (1). In Section 5 we show that u satisfies (1) iff

$$u(\mathbf{x}) = \frac{1}{8\pi^2}\sum_{i=1}^{n}\int_{Z_i} e^{i\boldsymbol{\mu}\cdot\mathbf{x}}\rho_i(\boldsymbol{\mu})\,d\nu_i(\boldsymbol{\mu}), \quad \mathbf{x} \in \Omega,$$

where each Z_i is a known subset of Z_P and the functions ρ_i and measures $d\nu_i$ are known *explicitly* in terms of the boundary values of u. These integral representations serve as a concrete realisation of the (abstract) fundamental principal of Ehrenpreis [7] which states that any solution to a constant coefficient PDE on a convex domain can be written as the superposition of exponential solutions.

The results in this paper serve to extend the highly acclaimed *unified method* of Fokas for elliptic boundary value problems [10]. This method has proved to be a highly efficient way of analysing constant coefficient elliptic equations on *planar* domains. The first steps in extending this method to three dimensional elliptic boundary value problems were produced in [3] for the special case of Laplace's equation. We build on these recent results to extend the unified method to *all* constant coefficient elliptic boundary value problems on three dimensional convex polyhedra. For boundary value problems for evolution PDEs in $2+1$ dimensions via the unified approach, we refer the reader to [9,11,15].

A note on generality The most general real, second order, constant coefficient elliptic differential operator has the form

$$\sum_{i,j=1}^{3} A_{ij}D_iD_j + \sum_{i=1}^{3} B_iD_i + C$$

where A_{ij} is symmetric and positive definite. By a simple linear change of coordinates (a rotation followed by a scaling of the coordinate axes), this operator can be written in the form

$$\sum_{i=1}^{3} \delta_{ij}D_iD_j + \sum_{i=1}^{3} b_iD_i + c$$