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On the spectrum of positive linear operators with a partition of unity property

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ABSTRACT

We characterize the spectrum of positive linear operators between Banach function spaces having finite rank and a partition of unity property. Our main result states that all the points in the spectrum are eigenvalues and 1 is the only eigenvalue on the unit circle. Finally, we show that the iterates converge in the uniform operator topology to a projection operator that reproduces constant functions and we provide a simple criterion to obtain the limiting projection operator.

We study positive linear operators that have finite rank on a general infinite-dimensional complex Banach function space X that contains the constant function 1 with norm equal to one. In addition, we assume that the associated basis functions of the positive finite-rank operator form a partition of unity. Operators of this kind are used in many applications to approximate functions where only a finite number of samples are available. The partition of unity property guarantees the exact reconstruction of constant functions. Of our interest here is the asymptotic behaviour of iterative applications of the operator and the question whether the limit exists.

The asymptotic behaviour of the iterates of positive linear operators has extensively been discussed by many authors. Kelisky and Rivlin [12] have first been considering the limit of iterates of the classical Bernstein operator on the space $C([0,1])$. This result has been extended by Karlin and Ziegler [10] to a more general setting. In [15,16], J. Nagel has examined the asymptotic behaviour of the Bernstein and the Kantorovič operators. Using a contraction principle, Rus [19] has shown an alternative way to prove the convergence of the iterates of the Bernstein operator. The iterates of the Bernstein operators have been also revisited by Badea [2] using spectral properties. Recently, contributions have been made by Gavrea and Ivan [4–7] and by Altomare [1] using methods based on Korovkin-type approximation theory. However, all these results are restricted to the space of continuous functions, i.e., are not applicable for the L^p spaces, and there is no general theory that guarantees the existence of the limit of the iterates.

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Here, we provide a functional analysis based approach using spectral properties that guarantees the existence of the limit of these iterates. We generalize some results of the manuscript [17], where spectral properties of this kind have been shown concretely for the variation-diminishing Schoenberg operator in order to show the limit of the iterates and to prove lower bounds for their approximation error in terms of different moduli of smoothness. If these spectral properties have been established, we apply the famous Theorem of Katznelson and Tzafriri [11] that states that the iterates converge in the operator norm if and only if the spectrum of T has either no points on the unit circle or 1 is the only spectral value on the unit circle. This article is devoted to give an application of this beautiful result in the field of approximation theory.

Finally, we remark that our results are applicable without explicit knowledge on spectral theory and additionally, we provide a simple criterion to derive the limiting projection operator.

1. Main results and examples

First, we introduce the notation used throughout this article. The general setting where our results are applicable is described next, followed by the presentation of our main results. We conclude this section with two examples to demonstrate the simplicity of our method.

1.1. Notation

Given two Banach spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and their topological duals $(X^*, \|\cdot\|_{X^*})$, $(Y^*, \|\cdot\|_{Y^*})$ respectively, we denote the space of bounded linear operators from X to Y by $\mathcal{B}(X, Y)$ equipped with the usual operator norm $\|\cdot\|_{op}$. With I we denote the identity operator on $\mathcal{B}(X, Y)$. For $T \in \mathcal{B}(X, Y)$, we denote by $\sigma(T)$ the spectrum of T and by $\sigma_p(T)$ the point spectrum of T, i.e., the set of all eigenvalues of T. We denote by $\mathcal{R}(T)$ the range and by $\mathcal{N}(T)$ the null space of T. The open unit disk in the complex plane will be denoted by $\mathbb{D}$ and its closure by $\overline{\mathbb{D}}$.

1.2. Setting

Let K be a compact Hausdorff space and let $(X, \|\cdot\|_X)$ be a complex infinite-dimensional Banach function space on K that contains the constant function 1 with $\|1\|_X = 1$. Given an integer $n > 0$ and linearly independent positive functions $e_1, \ldots, e_n \in X$ that form a partition of unity, i.e.,

$$\sum_{k=1}^{n} e_k = 1, \tag{1}$$

we set $Y := \operatorname{span}\{e_1, \ldots, e_n\}$. Clearly, Y is a finite-dimensional subspace of X with $1 \in Y$. Equipped with a norm $\|\cdot\|_Y$ that satisfies $\|1\|_Y = 1$, the space Y becomes a Banach space. Consider, e.g., Y equipped with the inherited norm of X.

Then we define the positive finite-rank operator $T : X \to Y$ by

$$Tf = \sum_{k=1}^{n} \alpha_k^*(f) e_k, \quad f \in X, \tag{2}$$

where α_k^* are positive linear functionals satisfying $\|\alpha_k^*\|_{X^*} = \alpha_k^*(1) = 1$ and $\alpha_k^*(e_k) > 0$ for $k \in \{1, \ldots, n\}$. There are many operators that match this definition, consider e.g., the Bernstein operator, Schoenberg's variation-diminishing spline operator, that arise in many applications in approximation theory and CAGD.

Finally, we remark that these operators mentioned previously are usually defined on a real Banach function space X. In this case we consider its complexification $X_{\mathbb{C}} = X \oplus iX$ equipped with the norm

$$\|f + ig\|_{\mathbb{C}} = \sup_{0 \le \varphi \le 2\pi} \|f \cos\varphi + g \sin\varphi\|_X, \quad f, g \in X.$$