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Proper orthogonal decomposition-based reduced-order stabilized mixed finite volume element extrapolating model for the nonstationary incompressible Boussinesq equations ☆

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A B S T R A C T

In this study, we employ a proper orthogonal decomposition (POD) method to establish a POD-based reduced-order stabilized mixed finite volume element (SMFVE) extrapolating model with very few degrees of freedom for the nonstationary incompressible Boussinesq equations. We provide error estimates of the POD-based reduced-order SMFVE solutions and the algorithm implementation for the POD-based reduced-order SMFVE extrapolating model. In addition, we present some numerical experiments that verify the validity and reliability of the POD-based reduced-order SMFVE extrapolating model.

1. Introduction

The nonstationary incompressible Boussinesq equations may be denoted by the following nonlinear system of partial differential equations (PDEs), including the velocity vector, pressure, and temperature (see [22,27,38]).

Problem I. Seek $\boldsymbol{u} = (u_1, u_2)^\tau$, p, and T such that, for $t_N > 0$,

$$\begin{cases} \boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = T\boldsymbol{j}, & (x,y,t) \in \Omega\times(0,t_N), \\ \nabla\cdot\boldsymbol{u} = 0, & (x,y,t) \in \Omega\times(0,t_N), \\ T_t - \gamma_0^{-1}\Delta T + (\boldsymbol{u}\cdot\nabla)T = 0, & (x,y,t) \in \Omega\times(0,t_N), \\ \boldsymbol{u}(x,y,t) = \boldsymbol{u}_0(x,y,t), \quad T(x,y,t) = \varphi(x,y,t), & (x,y,t) \in \partial\Omega\times(0,t_N), \\ \boldsymbol{u}(x,y,0) = \boldsymbol{u}^0(x,y), \quad T(x,y,0) = \psi(x,y) & (x,y) \in \Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^2$ is a bounded and interconnected domain, $\boldsymbol{u} = (u_1, u_2)^\tau$ denotes the fluid velocity vector, p is the pressure, T is the temperature, t_N is the total time, $\boldsymbol{j} = (0, 1)^\tau$ is the unit vector, $\nu = \sqrt{Pr/Re}$, Re is the Reynolds number, Pr is the Prandtl coefficient, and $\gamma_0 = \sqrt{Re\,Pr}$, $\boldsymbol{u}_0(x, y, t)$, $\boldsymbol{u}^0(x, y)$, $\varphi(x, y, t)$, and $\psi(x, y)$ all are known. For convenience and without any loss of generality, we may assume that $\boldsymbol{u}_0(x, y, t) = \boldsymbol{u}^0(x, y) = \mathbf{0}$ and $\varphi(x, y, t) = 0$ in the following theoretical study.

Because Problem I is a system of nonlinear PDEs, which includes the velocity vector, pressure, and temperature, there is usually no analytic solution. Thus, we have to rely on numerical solutions. The finite difference (FD) scheme (see, e.g., [38]), finite element (FE) technique (see, e.g., [11,22]), and the finite volume element (FVE) method (see, e.g., [21,27]) are three of the most common discrete methods for handling the nonstationary incompressible Boussinesq equations. However, compared with the FD and FE methods, the FVE method, which is also known as the generalized difference method (see [18,19]) or box method (see [3]), is regarded as the most effective discrete means for the nonstationary incompressible Boussinesq equations because it is usually easier to implement and it gives more flexibility when handling complex computing domains. In general, it can ensure local mass conservation and it has highly desirable properties in many applications. Therefore, it has also been used widely to search for numerical solutions of various types of PDEs (see, e.g., [2,3,5,7,9,10,13,16,18,19,35,39,40]).

The stabilized mixed finite volume element (SMFVE) technique for the nonstationary incompressible Boussinesq equations in [27] has more advantageous than its fully discrete FVE method without any stabilization in [21] (for example, it can avoid the constraint of the Brezzi–Babuška condition and its numerical solutions are more stable than those in [21]), but it includes many degrees of freedom (i.e., unknown quantities, which are the same as those in the fully discrete FVE method in [21]), thereby causing many difficulties in practical engineering computations, e.g., the computing load is very high and the accumulation of truncated errors in the computing process will increase very quickly. Therefore, an important problem is how to reduce the degrees of freedom for the fully discrete SMFVE formulation in order to alleviate the calculation load and the accumulation of truncated errors during the computing process, as well as reducing the time required for calculations in a manner that guarantees sufficiently accurate numerical solutions can be obtained.

The proper orthogonal decomposition (POD) technique (see [14,15]) is one of most effective tools for reducing the degrees of freedom in numerical models for time-dependent PDEs to alleviate the calculation load and the accumulation of truncated errors in the computing process. This technique has been used to establish some POD-based reduced-order Galerkin, FE, and FD numerical models for time-dependent PDEs (see [4,8,17,26,29,31–33,37]).

Some reduced FD schemes (see [23,36]) and mixed FE formulations (see [25,30]) based on the POD method have been established for the nonstationary incompressible Boussinesq equations but, to the best of our knowledge, no previous studies have used the POD technique to establish a POD-based reduced-order SMFVE extrapolating model for the nonstationary incompressible Boussinesq equations. A POD-based reduced-order FVE algorithm for viscoelastic equations (see [20]), a POD-based reduced-order Crank–Nicolson FVE formulation for parabolic equations (see [28]), and a reduced-order extrapolation algorithm based on the SFVE method and POD technique for non-stationary Stokes equations (see [24]) have been presented, but the nonstationary incompressible Boussinesq equations are far more complex than viscoelastic equations, parabolic equations, and nonstationary Stokes equations; thus, the POD-based reduced-order SMFVE extrapolating model for the nonstationary incompressible Boussinesq equations established in the present study has far more difficulties, which makes it more important, more serviceable, and more challenging than previous analyses. In particular, most existing POD-based reduced-order models (see, e.g., [4,8,17,20,26,25,28,29,31,30,32,33,37,36]) employ numerical solutions obtained from classical numerical models on the total time span $[0, t_N]$ to construct the POD bases and to establish POD-based reduced-order models, before recomputing the solutions on the same time span $[0, t_N]$. In fact, they involve repeated computations