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Generic continuity of metric entropy for volume-preserving diffeomorphisms ☆

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ABSTRACT

Let M be a compact manifold and $\mathrm{Diff}^1_m(M)$ be the set of C^1 volume-preserving diffeomorphisms of M. We prove that there is a residual subset $\mathcal{R} \subset \mathrm{Diff}^1_m(M)$ such that each $f \in \mathcal{R}$ is a continuity point of the map $g \to h_m(g)$ from $\mathrm{Diff}^1_m(M)$ to $\mathbb{R}$, where $h_m(g)$ is the metric entropy of g with respect to volume measure m.

1. Introduction

Let M be a smooth compact Riemannian manifold with dimension d, and m be a smooth volume measure on M. Without loss of generality, we always assume that $m(M) = 1$ in this paper. Denote by $\mathrm{Diff}^r_m(M)$ the set of C^r volume-preserving diffeomorphisms of M endowed with C^r topology for $r \geq 1$. A subset $\mathcal{R} \subset \mathrm{Diff}^r_m(M)$ is residual if it contains a countable intersection of C^r open and dense subsets of $\mathrm{Diff}^r_m(M)$.

Our main result is

Theorem 1.1. *There is a residual subset $\mathcal{R} \subset \mathrm{Diff}^1_m(M)$ such that each $f \in \mathcal{R}$ is a continuity point of the metric entropy map*

$$\mathcal{E} : \mathrm{Diff}^1_m(M) \to \mathbb{R}$$
$$g \mapsto h_m(g),$$

where $h_m(g)$ is the metric entropy of g with respect to volume measure m.

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The study of variation of entropy mainly focuses on two issues: the continuity of topological entropies and of metric entropies. In general, the variation of entropies is not even semicontinuous (e.g., see [9]). S. Newhouse [10] proved that the metric entropy function

$$\mu \to h_\mu(f)$$

is upper semicontinuous for all $f \in \mathrm{Diff}^\infty(M)$. In [14] (see also [6]), Y. Yomdin proved that the topological entropy function

$$f \to h(f)$$

is upper semicontinuous on $\mathrm{Diff}^\infty(M)$. Together with the result of A. Katok [7], topological entropy is continuous for C^∞ systems on surface. Recently, G. Liao et al. [8] extended the semicontinuity results of Newhouse and Yomdin to C^1 diffeomorphisms away from tangencies. The continuity of the metric entropy map $\mathcal{E}$ was considered firstly by A. Tahzibi in [13]. In fact, Tahzibi proved Theorem 1.1 in case $d = 2$ and we extend it to general case.

To prove Theorem 1.1, three main tools be used: Mañé–Bochi–Viana dichotomy [4], C^1 Pesin entropy formula [12,13] and some generic properties for volume-preserving diffeomorphisms [1]. There are some flow versions on above results (e.g., see [2,3]). So, we pose the following question

Question. Does the flows counterpart of Theorem 1.1 hold?

2. Preliminaries

2.1. Lyapunov exponents and dominated splitting

Given $f \in \mathrm{Diff}^1_m(M)$, by Oseledets Theory, there is an m-full invariant set $\mathcal{O} \subset M$ such that for every $x \in \mathcal{O}$ there exist a splitting (which is called *Oseledets splitting*)

$$T_xM = E_1(x) \oplus \cdots \oplus E_{k(x)}(x)$$

and real numbers (the *Lyapunov exponents* at x) $\chi_1(x,f) > \chi_2(x,f) > \cdots > \chi_{k(x)}(x,f)$ satisfying $Df(E_j(x)) = E_j(f(x))$ and

$$\lim_{n\to\pm\infty} \frac{1}{n} \ln\|Df^n v\| = \chi_j(x,f)$$

for every $v \in E_j(x) \setminus \{0\}$ and $j = 1, 2, \cdots, k(x)$. In the following, by counting multiplicity, we also rewrite the Lyapunov exponents of m as

$$\lambda_1(x,f) \geq \lambda_2(x,f) \geq \cdots \geq \lambda_d(x,f).$$

For $x \in \mathcal{O}$, we denote by

$$\xi_i(x,f) = \begin{cases} \lambda_i(x,f), & \text{if } \lambda_i(x,f) \geq 0; \\ 0, & \text{if } \lambda_i(x,f) < 0 \end{cases}$$

and

$$\chi^+(x,f) = \sum_{i=1}^{d} \xi_i(x,f).$$