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Logarithmic Harnack inequalities for homogeneous graphs

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A B S T R A C T

In this paper, we prove logarithmic Harnack inequalities for homogeneous graphs. As a consequence, we derive lower estimates for the log-Sobolev constant, extending previous results for Ricci flat graphs.

1. Introduction

Suppose G is a graph with vertex set V and edge set E. The degree of vertex x, denoted by d_x, is the number of edges connected to x. If for every vertex x of V, d_x is finite, we say that G is a locally finite graph. The distance between two vertices is the minimum number of edges to connect them, while the diameter of G is the maximum of all the distances of the graph. We denote $x \sim y$ if vertex x is adjacent to vertex y, and μ_{xy} is the edge weight. Moreover, suppose a group χ acts on V such that:

(i) for all $a \in \chi$, $\{au, av\} \in E$ if and only if $\{u, v\} \in E$;
(ii) for any two vertices u and v, there is an $a \in \chi$ such that $au = v$.

Then we say G is a homogeneous graph with the associated group χ. Furthermore, we describe the edge set by an edge generating set $K \subset \chi$, then, for some $v \in V$ and $a \in K$, each edge of G is of the form $\{v, av\}$. We let K consist of k generators and require K to be symmetric, i.e., $a \in K$ if and only if $a^{-1} \in K$. If for every element $a \in K$, we have $aKa^{-1} = K$, we say that a homogeneous graph is invariant. If χ is abelian, we say G is an abelian homogeneous graph.

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Let $V^R = \{f \mid f : V \to R\}$, and the Laplace operator $\mathcal{L}$ of a graph G be

$$\mathcal{L}f(x) = \frac{1}{k}\sum_{a\in K}\big[f(x) - f(ax)\big], \quad \forall f \in V^R.$$

Suppose a function $f : V \to R$ satisfies $\mathcal{L}f(x) = \lambda f(x)$, then f is called an eigenfunction of Laplace operator $\mathcal{L}$ on graph G with eigenvalue λ, and we can easily note that 0 is a trivial eigenvalue of $\mathcal{L}$ associated with the constant eigenfunction.

According to Bakry and Emery [1], we can define a bilinear operator $\Gamma : V^R \times V^R \to V^R$ by

$$\Gamma(f,g)(x) = \frac{1}{2}\big\{f(x)\mathcal{L}g(x) + g(x)\mathcal{L}f(x) - \mathcal{L}\big(f(x)g(x)\big)\big\},$$

and then the Ricci curvature operator on graphs Γ_2 by iterating Γ as

$$\Gamma_2(f,g)(x) = \frac{1}{2}\big\{\Gamma(f,\mathcal{L}g)(x) + \Gamma(g,\mathcal{L}f)(x) - \mathcal{L}\Gamma(f,g)(x)\big\}.$$

More explicitly, we have

$$\rho(x) = \frac{1}{k}\sum_{a\in K}\big[f(x) - f(ax)\big]^2,$$

$$\Gamma(f,f)(x) = \frac{1}{2}\rho(x) = \frac{1}{2}\cdot\frac{1}{k}\sum_{a\in K}\big[f(x) - f(ax)\big]^2.$$

Definition 1.1. The operator $\mathcal{L}$ satisfies the curvature-dimension type inequality $CD(m,\xi)$ for some $m > 1$ if for every $f \in V^R$,

$$\Gamma_2(f,f)(x) \geq \frac{1}{m}\big(\mathcal{L}f(x)\big)^2 + \xi\Gamma(f,f)(x).$$

We call m the dimension of the operator $\mathcal{L}$ and ξ the lower bound of the Ricci curvature of the operator $\mathcal{L}$. It is easy to see that for $m < \tilde{m}$, the operator $\mathcal{L}$ satisfies $CD(\tilde{m},\xi)$ if it satisfies $CD(m,\xi)$.

If $\Gamma_2(f,f)(x) \geq \xi\Gamma(f,f)(x)$, we say that $\mathcal{L}$ satisfies $CD(\infty,\xi)$.

In 1994, F. Chung and S.T. Yau in [4] established the following Harnack inequality as in [2] for homogeneous graphs and subgraphs G with edge generating set K consisting of k generators,

$$\frac{1}{k}\sum_{a\in K}\big[f(x) - f(ax)\big]^2 + \alpha\lambda f^2(x) \leq \frac{\lambda\alpha^2}{\alpha - 2}\sup_{y\in K} f^2(y)$$

for any $\alpha > 2$ and $x \in V$, and using this Harnack inequality, they derived lower bounds for the Neumann eigenvalues and the Dirichlet eigenvalues in [4] and [6] respectively.

In 1996, F.R.K. Chung and S.T. Yau in [5] proved the logarithmic Harnack inequality for Ricci flat graphs. In fact, a homogeneous graph associated with an abelian group is Ricci flat such as in [5]. Furthermore, they derived a lower bound for the log-Sobolev constant of Ricci flat graphs using the logarithmic Harnack inequality.

In 2010, Y. Lin and S.T. Yau introduced in [7] the curvature-dimension type inequality $CD(m,\kappa)$ and proved that any locally finite connected graph satisfies either $CD(2,\frac{2}{d}-1)$ if d is finite, or $CD(2,-1)$ if d is infinite, where $d = \sup_{x\in V}\sup_{y\sim x}\frac{d_x}{\mu_{xy}}$. They also proved that the Ricci flat graphs have the non-negative Ricci curvature in the sense of Bakry and Emery. In fact, in most cases, the Ricci curvature is zero, except