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Oscillation and integral norms of coefficients in second-order differential equations

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ABSTRACT

By theorems of Sturm and Lyapunov, the L^∞ and L^1 norms of the coefficient q in a differential equation $u'' + q(t)u = 0$ limit the oscillation of nontrivial solutions. This paper establishes similar results for the L^p norms when $1 < p < \infty$.

For real differential equations $u'' + q(t)u = 0$, the size of the coefficient function limits the frequency with which nontrivial solutions can vanish. Classical instances of that principle include the following (see pp. 334 and 346 of [4]):

Theorem *(Sturm, Lyapunov). If q is continuous and $a < b$ are zeroes of a nontrivial solution of $u'' + q(t)u = 0$, then*

$$\max_{t\in[a,b]} q(t) \geq \frac{\pi^2}{(b-a)^2}, \qquad \int_a^b \max\{q(t), 0\}\, dt > \frac{4}{b-a}.$$

The former bound is attained when $u(t) = \sin(\pi(t-a)/(b-a))$. Lyapunov's bound is also optimal, as Theorem 9 in Section 4 will confirm. Further instances of the principle appear in [2] and [1].

Under the stated hypotheses, the L^∞ and L^1 norms of q in $[a,b]$ satisfy

$$\|q\|_\infty := \max_{t\in[a,b]} |q(t)| \geq \frac{\pi^2}{(b-a)^2}, \qquad \|q\|_1 := \int_a^b |q(t)|\, dt > \frac{4}{b-a}.$$

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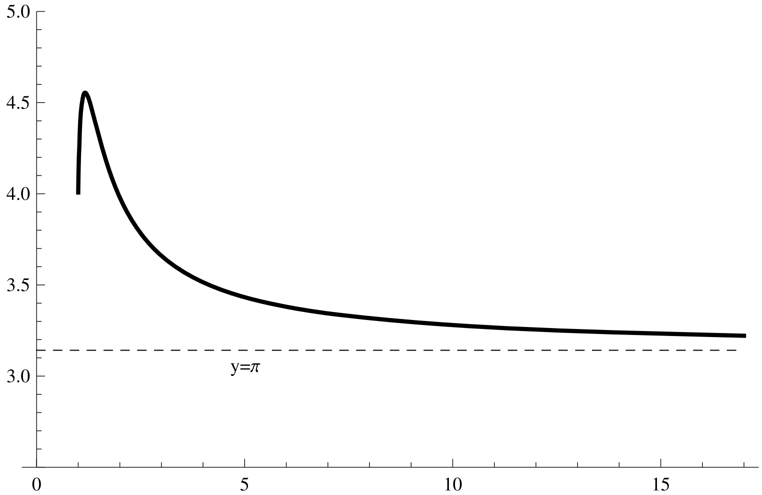

Fig. 1. The function β.

The present paper establishes similar bounds for the L^p norms when $1 < p < \infty$. The essential question here is: If a continuous function q in $[0, \infty)$ satisfies $\int |q|^p \leq 1$ and u is the solution of $u'' + qu = 0$ with $(u, u')(0) = (0, 1)$, how small can the first positive zero of u be? It can certainly be less than 10, regardless of p, as one sees by considering functions q that equal $1/\pi^2$ throughout $[0, \pi^2]$ and drop rapidly to zero thereafter. We therefore work in $[0, 10]$ rather than $[0, \infty)$. We also replace the initial-value problem with the equivalent integral equation

$$u(t) + \int_0^t (t - s)q(s)u(s)\, ds = t, \quad t \in [0, 10],$$

and allow q to range over the closed unit ball in $L^p[0, 10]$. As will be shown, the integral equation makes sense even then as a condition on $u \in C[0, 10]$ and has a unique solution $u = u_q$ in that space. (See Fig. 1.)

Theorem 1. *Let* $W = \{q \in L^p[0, 10] : \|q\|_p \leq 1\}$*, where* $1 < p < \infty$*. The infimum* $\beta = \inf\{z \in (0, 10] : u_q(z) = 0$ *for some* $q \in W\}$ *is given by*

$$\beta = \beta(p) := \left\{ \frac{(2p-1)^{2p-1}}{p^p (p-1)^{(p-1)}} \cdot B\left(\frac{3p-1}{2p}, \frac{1}{2}\right)^{2p} \right\}^{1/(2p-1)},$$

where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$*. Furthermore, there is a unique element* $q \in W$ *such that* $u_q(\beta) = 0$*. That element is represented by a continuous function that is positive throughout* $(0, \beta)$ *and zero elsewhere, and* $\|q\|_p = 1$*.*

Identities such as $x\Gamma(x) = \Gamma(x + 1)$ and $B(x, \frac{1}{2}) = 2^{2x-1}B(x, x)$ (see §9.2–9.4 of [3]) support other formulas for $\beta(p)$, but the one here seems as simple and informative as any.

Corollary 2. *If* q *is continuous and* $a < b$ *are zeroes of a nontrivial solution of* $u'' + q(t)u = 0$*, then the norms* $\|q\|_p = (\int_a^b |q(t)|^p\, dt)^{1/p}$ *for* $1 < p < \infty$ *satisfy*

$$\|q\|_p \geq \left(\frac{\beta(p)}{b - a}\right)^{(2p-1)/p} = \frac{(2p-1)^{(2p-1)/p}}{p(p-1)^{(p-1)/p}} \cdot \frac{B((3p-1)/(2p), 1/2)^2}{(b-a)^{(2p-1)/p}}.$$