J. Math. Anal. Appl. 425 (2015) 460–488

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Independent products in infinite spaces

Enrique Miranda [a,*], Marco Zaffalon [b]

[a] *University of Oviedo, Department of Statistics and Operations Research, C-Calvo Sotelo, s/n, 33007 Oviedo, Spain*
[b] *Istituto Dalle Molle di Studi sull'Intelligenza Artificiale (IDSIA), Galleria 2, 6928 Manno (Lugano), Switzerland*

ARTICLE INFO

Article history:
Received 19 February 2014
Available online 23 December 2014
Submitted by U. Stadtmueller

Keywords:
Independence
Irrelevance
Imprecise probability
Coherence
Infinite spaces
Factorisation

ABSTRACT

Probabilistic independence, intended as the mutual irrelevance of given variables, can be solidly founded on a notion of self-consistency of an uncertainty model, in particular when probabilities go imprecise. There is nothing in this approach that prevents it from being adopted in very general setups, and yet it has mostly been detailed for variables taking finitely many values. In this mathematical study, we complement previous research by exploring the extent to which such an approach can be generalised. We focus in particular on the independent products of two variables. We characterise the main notions, including some of factorisation and productivity, in the general case where both spaces can be infinite and show that, however, there are situations—even in the case of precise probability—where no independent product exists. This is not the case as soon as at least one space is finite. We study in depth this case at the frontiers of good-behaviour detailing the relations among the most important notions; we show for instance that being an independent product is equivalent to a certain productivity condition. Then we step back to the general case: we give conditions for the existence of independent products and study ways to get around its inherent limitations.

1. Introduction

Independence is a founding notion for very many probabilistic models and applications. Despite its widespread use, there are substantial aspects of this notion that are still troublesome, in particular when we stick to the traditional approach to defining independence—the one based on requiring that a joint probability factorises. To make an example, if we say that events A and B are independent when $P(A \cap B) = P(A)P(B)$, what we get is that A is independent of any other event B if $P(A) = 0$, *including the event* $B = A^c$! This happens because the traditional definition neglects the issues originated by zero probabilities. This affects the case of finite spaces of possibility and even more the models based on infinite spaces, where

* Corresponding author.
E-mail addresses: mirandaenrique@uniovi.es (E. Miranda), zaffalon@idsia.ch (M. Zaffalon).

http://dx.doi.org/10.1016/j.jmaa.2014.12.049
0022-247X/

it is very common that each element in the space has zero probability. The situation becomes more complex when we consider models that allow for imprecisely specified probabilities. In this case an event A has both a lower probability $\underline{P}(A)$ and an upper probability $\overline{P}(A)$, so it can happen that $\underline{P}(A) = 0 < \overline{P}(A)$. It is clear that in this case we should have a uniform way to deal with independence that works irrespective of the positivity of probabilities. The imprecise case poses a series of other challenges as well: it has been shown that there are many possible definitions of independence in such a generalised setup [2,7]; very often imprecise probability models are made of sets of finitely additive probabilities, which create additional complications w.r.t. the more regular countably additive probabilities.

It has been Walley to illustrate [24, Chapter 9] that all these issues can be nicely and uniformly addressed by a shift of paradigm in the way we define independence. Walley's approach is based on joining two pre-existing ideas. The first, which has its roots in the subjective approach to probability, as well as in the artificial intelligence community, is regarding independence as the mutual irrelevance of two events (or variables). The second, especially due to de Finetti, is that probabilistic models can be founded on a notion of self-consistency—most often called *coherence*. In this paper we shall, for the most part, refer to Walley's coherence notion [24, Section 7.1.4(b)]. We can think of coherence as a stronger way to define probabilistic models than through a joint distribution. In fact, the existence of a joint distribution compatible with some marginal and conditional distributions, can equivalently be formulated as a self-consistency requirement, which is however weaker than coherence (see, e.g., [17, Section 4]). This weakness shows up dramatically just when there are events with zero probability, when we work in infinite spaces, when we deal with finitely additive probabilities, and of course also when we deal with imprecise probability. This is the reason why the stronger notion of coherence is at the basis of a more powerful approach to defining independence.

So how do we formulate independence through coherence? Consider variables X_1 and X_2, taking values from $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively. We just say that a joint probabilistic model for these variables is an *independent product* of the marginal information we have about X_1 and X_2 if it is coherent with our assessment that knowledge of one of the two variables does not affect knowledge about the other variable. It is an independent product, in other words, if it is not inconsistent with two assessments of irrelevance: the irrelevance of X_1 to X_2 and of X_2 to X_1. Given that coherence is a notion defined under very general assumptions, we have automatically a well-posed way to discuss the notion of independence across all the situations mentioned above. As we said, approaching independence from the point of view of coherence has its roots in Walley's seminal book, which laid down the main ideas. Vicig [22] then pursued similar ideas using a weaker notion of coherence by Williams [25] (we call it *W-coherence*); his work addressed the case of infinite spaces of possibility, while restricting the attention to the special case of lower probabilities rather than expectations (or, as we call them, *previsions*). Finally, de Cooman and ourselves [10,12] have studied the case of finite spaces of possibility, while allowing for more than two variables; the work focused in particular on the least-committal (or least precise, or weakest) independent product, which is called the *independent natural extension*, and on ways to relate it to different forms of factorisation. The independent natural extension is an important concept as it is the only independent product that is solely based on the mutual irrelevance of the variables under consideration.

In this paper, we analyse the independent products of two variables aiming at the greatest possible generality—thus covering the case of infinite spaces of possibility—as well as at establishing firm relations with known notions of factorisation—thus investigating the extent to which the traditional notion of independence and the notion based on coherence are compatible. After some preliminary concepts are given in Section 2, we start work on independent products in Section 3. First, we remark by an example that two given marginals may not admit *any* independent product when both spaces $\mathcal{X}_1, \mathcal{X}_2$ are infinite, not even in the case of precise probability: this shows that there are limits to the possibility to define independence in the general case. As a consequence, the independent natural extension may not exist either, unlike the case of finite spaces [12]. Yet, when it exists, we show that it can be characterised as the intersection of the two sets of probabilities that express irrelevance of X_1 to X_2 and of X_2 to X_1: this is a remarkably simple