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Complete characterization of Hadamard powers preserving Loewner positivity, monotonicity, and convexity ☆

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A B S T R A C T

Entrywise powers of symmetric matrices preserving positivity, monotonicity or convexity with respect to the Loewner ordering arise in various applications, and have received much attention recently in the literature. Following FitzGerald and Horn (1977) [8], it is well-known that there exists a *critical exponent* beyond which all entrywise powers preserve positive definiteness. Similar phenomena have also recently been shown by Hiai (2009) to occur for monotonicity and convexity. In this paper, we complete the characterization of all the entrywise powers below and above the critical exponents that are positive, monotone, or convex on the cone of positive semidefinite matrices. We then extend the original problem by fully classifying the positive, monotone, or convex powers in a more general setting where additional rank constraints are imposed on the matrices. We also classify the entrywise powers that are super/sub-additive with respect to the Loewner ordering. Finally, we extend all the previous characterizations to matrices with negative entries. Our analysis consequently allows us to answer a question raised by Bhatia and Elsner (2007) regarding the smallest dimension for which even extensions of the power functions do not preserve Loewner positivity.

1. Introduction and main results

The study of positive definite matrices and of functions that preserve them arises naturally in many branches of mathematics and other disciplines. Given a function $f : \mathbb{R} \to \mathbb{R}$ and a matrix $A = (a_{ij})$, the matrix $f[A] := (f(a_{ij}))$ is obtained by applying f to the entries of A. Such mappings are called *Hadamard functions* (see [16, §6.3]) and appear naturally in many fields of pure and applied mathematics, probability, and statistics.

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Characterizing functions $f : \mathbb{R} \to \mathbb{R}$ such that $f[A]$ is positive semidefinite for every positive semidefinite matrix A is critical for many applications. For example, in modern high-dimensional probability and statistics, functions are often applied to the entries of covariance/correlation matrices in order to obtain regularized estimators with attractive properties (like sparsity, good condition number, etc.). Particular examples of functions used in practice include the so-called *hard* and *soft-thresholding* functions (see [5, 11–14]), and the power functions – see e.g. [20] and [24, §2.2]. The resulting matrices often serve as ingredients in other statistical procedures that require these matrices to be positive semidefinite. In order for such procedures to be widely applicable, it is therefore important to know whether a given Hadamard function preserves positivity.

Let A be a positive semidefinite matrix with nonnegative entries. In this paper, we study the properties of entrywise powers of A, i.e., the properties of $f[A]$ when $f(x) = x^\alpha$ is applied elementwise to A (for some $\alpha \geq 0$). This question of which entrywise (or Hadamard) powers x^α preserve Loewner positivity has been widely studied in the literature. One of the earliest works in this setting is by FitzGerald and Horn [8], who studied the set of entrywise powers preserving Loewner positivity among $n \times n$ matrices, in connection with the Bieberbach conjecture. They show that a certain phase transition occurs at $\alpha = n - 2$. More precisely, every $\alpha \geq n - 2$ as well as every positive integer preserve Loewner positivity, while no non-integers in $(0, n - 2)$ do so.

The phase transition at the integer $n - 2$ has been popularly referred to in the literature as the "critical exponent" (CE) for preserving Loewner positivity. (We remark that the notion of critical exponents in this paper differs from that in the physics literature, where it is used in the context of many-body systems.) Indeed, the study of critical exponents in the present context – and more generally of functions preserving a form of positivity – is an interesting and important endeavor in a wide variety of situations, and has been studied in many settings (see e.g. [21,17,10,22,18]).

While it is more common in the critical exponents literature to study matrices with nonnegative entries, positive semidefinite matrices containing negative entries also occur frequently in practice. In recent work, Hiai [15] extended previous work by FitzGerald and Horn by considering the odd and even extensions of the power functions to $\mathbb{R}$. Recall that for $\alpha \in \mathbb{R}$, the even and odd multiplicative extensions to $\mathbb{R}$ of the power function $f_\alpha(x) := x^\alpha$ are defined to be $\phi_\alpha(x) := |x|^\alpha$ and $\psi_\alpha(x) := \operatorname{sign}(x)|x|^\alpha$ at $x \neq 0$, and $f_\alpha(0) = \phi_\alpha(0) = \psi_\alpha(0) := 0$. In [15], Hiai studied the powers $\alpha > 0$ for which ϕ_α and ψ_α preserve Loewner positivity, and showed that the same phase transition also occurs at $n - 2$ for ϕ_α, ψ_α, as demonstrated in [8]. He also analyzed functions that are *monotone* and *convex* with respect to the Loewner ordering, and proved several deep results and connections between these classes of functions. These results are akin to the corresponding connections between positivity, monotonicity, and convexity for real functions of one variable. Before recalling these notions, we first introduce some notation. Given $n \in \mathbb{N}$ and $I \subset \mathbb{R}$, let $\mathbb{P}_n(I)$ denote the set of symmetric positive semidefinite $n \times n$ matrices with entries in I; denote $\mathbb{P}_n(\mathbb{R})$ by $\mathbb{P}_n$. We write $A \geq B$ when $A - B \in \mathbb{P}_n$. For a function $f : I \to \mathbb{R}$ and a matrix $A \in \mathbb{P}_n(I)$, we denote by $f[A]$ the matrix $f[A] := (f(a_{ij}))$. For a matrix A with nonnegative entries, the entrywise power $A^{\circ\alpha} := ((a_{ij}^\alpha))$ then equals $f_\alpha[A]$. Given a subset $V \subset \mathbb{P}_n(I)$, recall [15] that a function $f : I \to \mathbb{R}$ is

- *positive on* V with respect to the Loewner ordering if $f[A] \geq 0$ for all $0 \leq A \in V$;
- *monotone on* V with respect to the Loewner ordering if $f[A] \geq f[B]$ for all $A, B \in V$ such that $A \geq B \geq 0$;
- *convex on* V with respect to the Loewner ordering if $f[\lambda A + (1 - \lambda)B] \leq \lambda f[A] + (1 - \lambda)f[B]$ for all $0 \leq \lambda \leq 1$ and $A, B \in V$ such that $A \geq B \geq 0$;
- *super-additive on* V with respect to the Loewner ordering if $f[A + B] \geq f[A] + f[B]$ for all $A, B \in V$ for which $f[A + B]$ is defined;
- *sub-additive on* V with respect to the Loewner ordering if $f[A + B] \leq f[A] + f[B]$ for all $A, B \in V$ for which $f[A + B]$ is defined.