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Surjective isometries on the vector-valued differentiable function spaces

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ABSTRACT

This paper investigates the surjective linear isometries between the differentiable function spaces $C_0^n(\Omega, E)$ and $C_0^m(\Sigma, F)$ (where Ω, Σ are open subsets of Euclidean spaces and E, F are reflexive, strictly convex Banach spaces), and show that such isometries can be written as weighted composition operators.

1. Introduction

The classical Banach–Stone Theorem gives a characterization of surjective linear isometries between the spaces of scalar-valued continuous functions on compact Hausdorff spaces. There are several extensions of the Banach–Stone Theorem to a variety of different settings (for a survey of this topic we refer the readers to the monographs [7,8]). M. Cambern and V.D. Pathak [3,4] considered surjective linear isometries on spaces of scalar-valued differentiable functions on the locally compact subset of $\mathbb{R}$ containing no isolated points and gave a representation of such mappings; and then Jarosz and Pathak [10] studied those operators on differentiable function spaces defined on the compact subset X of a real line (they did not assume X contains no isolated point). Recently, F. Botelho and J. Jamison [2] extended these results to some vector-valued differentiable function space $C^1([0, 1], H)$, where H is a finite-dimensional Hilbert space. In their proof, the characterization of extreme points of the unit dual ball plays a crucial role. The second author [11] gave the representation of surjective linear isometries on the scalar-valued differentiable function spaces $C_0^n(X)$, where X is an open subset of Euclidean spaces $\mathbb{R}^p$ satisfying some other property (which is called NIP).

In this paper, we will give a characterization of the surjective linear isometries between the spaces of vector-valued differentiable functions $C_0^n(\Omega, E)$, where Ω is an open subset of $\mathbb{R}^p$ and E is a reflexive and

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strictly convex Banach space. This will extend the main results of [2–4,11] and give the Banach–Stone Theorem on the differentiable function spaces.

Let $\mathbb{Z}_+$ be the set of non-negative integers. When $p \in \mathbb{N}$ and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p)$ is a p-tuple in $\mathbb{Z}_+$, we set $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_p$ and $\lambda! = \lambda_1!\lambda_2! \cdots \lambda_p!$. Suppose that Ω is an open subset of $\mathbb{R}^p$, then $C_0^n(\Omega, E)$ denotes the space consisting of all the E-valued functions on Ω that are vanishing at infinity and of class C^n, that is, those functions whose partial derivatives

$$\partial^\lambda f = \frac{\partial^{\lambda_1 + \cdots + \lambda_p} f}{\partial x_1^{\lambda_1} \partial x_2^{\lambda_2} \cdots \partial x_p^{\lambda_p}}$$

exist and are continuous for every $\lambda = (\lambda_1, \cdots, \lambda_p) \in \Lambda$, where $\Lambda = \{\lambda \in \mathbb{Z}_+^p : |\lambda| \le n\}$. The norm in the Banach space $C_0^n(\Omega, E)$ is defined by

$$\|f\| = \sup_{x \in \Omega} \sum_{\lambda \in \Lambda} \frac{\|\partial^\lambda f(x)\|}{\lambda!}, \quad \forall\, f \in C_0^n(\Omega, E).$$

For any $f \in C_0^n(\Omega, E)$ and $g \in C_0^n(\Omega)$, it is easy to verify that $C_0^n(\Omega, E)$ is a $C_0^n(\Omega)$-module, that is, $fg \in C_0^n(\Omega, E)$ and $\|fg\| \le \|f\|\|g\|$. For any scalar-valued function $f \in C_0^n(\Omega)$ and any element $e \in E$, the vector-valued function $x \in \Omega \mapsto f(x)e$ is denoted by $f \otimes e$. For any (scalar-valued or vector-valued) function f, supp(f) is denoted by the closure of cozero set of f, that is, $\mathrm{supp}(f) = \overline{\{x : f(x) \neq 0\}}$.

2. Extreme points of the unit dual ball of $C_0^n(\Omega, E)$

In this section, we will give the complete characterization of the extreme points in the dual ball of the spaces of differentiable functions. Recall the following construction of differentiable functions due to the second author [11].

Lemma 2.1. *(See [11, Proposition 1.1].) For any $x_0 \in \mathbb{R}^p$ and $\varepsilon, \delta > 0$, there exists a function $f \in C_0^n(\mathbb{R}^p)$ such that $\|f\| = 1$, supp$(f) \subset B(x_0, \delta)$ and*

$$\frac{1}{n!}\left|\frac{\partial^{(n,0,\cdots,0)} f(x_0)}{\partial x_1^n}\right| > 1 - \varepsilon.$$

We denote by B_E the closed unit ball of a Banach space E. Equipped B_{E^*} with the weak*-topology, one can isometrically embed $C_0^n(\Omega, E)$ onto a subspace $\mathcal{M}$ of $C_0(W)$ (in this case, we can identify $C_0^n(\Omega, E)$ with $\mathcal{M}$), the space of continuous functions in $W = \Omega \times (B_{E^*})_{\lambda \in \Lambda}$ that vanishes at infinity, by $f \in C_0^n(\Omega, E) \mapsto \tilde{f} \in C_0(W)$, where $\tilde{f}$ is defined by

$$\tilde{f}(x, (\varphi_\lambda)_{\lambda \in \Lambda}) = \sum_{\lambda \in \Lambda} \frac{\varphi_\lambda(\partial^\lambda f(x))}{\lambda!}, \quad \forall\, (x, (\varphi_\lambda)_{\lambda \in \Lambda}) \in W.$$

Analogous to [2, Proposition 1.3] and [11, Lemma 1.2], one can derive the following characterization of extreme points of the unit dual ball of $C_0^n(\Omega, E)$.

Theorem 2.2. *Suppose that E is a reflexive and strictly convex Banach space. Then Φ is an extreme point of the unit dual ball $B_{\mathcal{M}^*}$ if and only if there exists $(x, \tilde{\varphi}) \in W$, with $\tilde{\varphi} = (\varphi_\lambda)_{\lambda \in \Lambda}$ and all φ_λ are extreme points of B_{E^*}, such that*

$$\Phi(f) = \sum_{\lambda \in \Lambda} \frac{\varphi_\lambda(\partial^\lambda f(x))}{\lambda!}, \quad \forall\, f \in C_0^n(\Omega, E). \tag{2.1}$$