J. Math. Anal. Appl. 427 (2015) 741–758

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Rational period functions and cycle integrals in higher level cases

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ARTICLE INFO

Article history:
Received 25 November 2014
Available online 24 February 2015
Submitted by B.C. Berndt

Keywords:
Rational period functions
Period polynomials
Weakly holomorphic modular forms
Cycle integrals

ABSTRACT

Generalizing the results of Duke, Imamoḡlu, and Tóth [8] we give an effective basis for the space of period polynomials in higher level cases and construct modular integrals for certain rational period functions related to indefinite binary quadratic forms by means of cycle integrals.

1. Introduction and statement of results

Let k be an integer. A rational function $\psi(z)$ that satisfies

$$\psi(z) = \psi(1+z) + (z+1)^{k-2}\psi\left(\frac{z}{z+1}\right) \tag{1}$$

is called a *period function*. A *modular integral* for ψ is defined as a holomorphic function F on $\mathfrak{H}$ (= complex upper half plane) with $F(z+1) = F(z)$ and meromorphic at the cusp ∞ that satisfies

$$\psi(z) = F(z) - z^{k-2}F(-1/z).$$

The notion of rational period functions was initiated by Knopp [13] and he showed that an arbitrary rational period function can have poles only at 0 or at real quadratic irrationalities. Since Knopp's work, further theory has been developed by many authors (see the references in [5, Section 2.4]).

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[1] Choi was supported by the Dongguk University Research Fund of 2013 and by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012R1A1A3011711).
[2] Kim was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2013R1A1A2A10004632).

http://dx.doi.org/10.1016/j.jmaa.2015.02.069
0022-247X/

For any meromorphic function f on $\mathfrak{H}$, we define the action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$ by

$$(f|_k\gamma)(z) = (cz+d)^{-k} f(\gamma z).$$

For even $k > 2$, let W_{k-2} be the space of polynomials that satisfy (1). Equivalently, W_{k-2} can be defined as the space of all polynomials ψ of degree at most $k-2$ that satisfy

$$\psi + \psi|_{2-k}S = \psi + \psi|_{2-k}TS + \psi|_{2-k}(TS)^2 = 0$$

with $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The elements of the space W_{k-2} are called *period polynomials*. Period polynomials have been investigated in relation to modular integrals, cusp forms via Eichler–Shimura isomorphism and to various other areas of mathematics (see [12,14,17]). The importance of period polynomials comes from their close connection with special values of modular L-functions.

In [8] Duke, Imamoḡlu, and Tóth gave an effective basis for the space W_{k-2} using weakly holomorphic modular forms. Furthermore for any integer k they constructed modular integrals $F(z, Q)$ for certain rational solution ψ_Q to (1), which comes from indefinite binary quadratic forms Q. The aim of this paper is to extend their results to higher level cases.

Let p be one or a prime and $\Gamma_0^+(p)$ be the group generated by the Hecke group $\Gamma_0(p)$ and the Fricke involution $W_p = \left(\begin{smallmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{smallmatrix}\right)$. Let k be a positive integer greater than 2 and let P_{k-2} denote the space of all polynomials of degree at most $k-2$. For $p \in \{2,3\}$ we define a subspace W_{k-2}^+ of P_{k-2} by

$$W_{k-2}^+ = \{g \in P_{k-2} \mid g + g|_{2-k}W_p = 0 = g + g|_{2-k}U + g|_{2-k}U^2 + \cdots + g|_{2-k}U^{2p-1}\}$$

with $U = TW_p$. Let $S_k(\Gamma_0^+(p))$ be the space of cusp forms (that is, modular forms which vanish at the cusp ∞) of weight k for $\Gamma_0^+(p)$.

Theorem 1.1. *Let $p \in \{2,3\}$.*
(i) The dimension of the subspace W_{k-2}^+ is given by

$$\dim W_{k-2}^+ = 2\left(\left\lceil \frac{k-2}{4} \right\rceil - \left\lceil \frac{\frac{k}{2}-1}{2p} \right\rceil\right) - 1,$$

where $\lceil x \rceil$ means the least integer greater than or equal to x.
(ii) We have

$$\dim W_{k-2}^+ = 2\dim S_k(\Gamma_0^+(p)) + 1.$$

For any even integer k let $M_k^!(\Gamma_0^+(p))$ be the space of weakly holomorphic modular forms (that is, meromorphic with poles only at the cusp ∞) of weight k for $\Gamma_0^+(p)$. Each $f \in M_k^!(\Gamma_0^+(p))$ has a Fourier expansion of the form

$$f(z) = \sum_{n \geq n_0} a_f(n) e(nz),$$

where $e(z) = \exp(2\pi i z)$. We set $\mathrm{ord}_\infty f = n_0$ if $a_f(n_0) \neq 0$. When the genus of $\Gamma_0^+(p)$ is zero, the space $M_k^!(\Gamma_0^+(p))$ has a canonical basis [6,3,2]. Let m_k denote the maximal order of a nonzero $f \in M_k^!(\Gamma_0^+(p))$ at ∞. For every integer $m \geq -m_k$, there exists a unique weakly holomorphic modular form $f_{k,m} \in M_k^!(\Gamma_0^+(p))$ with Fourier expansion of the form

$$f_{k,m}(z) = q^{-m} + \sum_{n > m_k} a_k(m,n) q^n \quad (q = e(z))$$

and together they form a basis for $M_k^!(\Gamma_0^+(p))$.