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On the norm of products of polynomials on ultraproducts of Banach spaces [☆]

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ABSTRACT

The purpose of this article is to study the problem of finding sharp lower bounds for the norm of the product of polynomials in the ultraproducts of Banach spaces $(X_i)_{\mathfrak{U}}$. We show that, under certain hypotheses, there is a strong relation between this problem and the same problem for the spaces X_i.

1. Introduction

In this article we study the factor problem in the context of ultraproducts of Banach spaces. This problem can be stated as follows: for a Banach space X over a field $\mathbb{K}$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and natural numbers $k_1, \cdots, k_n$ find the optimal constant M such that, given any set of continuous scalar polynomials $P_1, \cdots, P_n : X \to \mathbb{K}$, of degrees $k_1, \cdots, k_n$; the inequality

$$M\|P_1 \cdots P_n\| \geq \|P_1\| \cdots \|P_n\| \tag{1}$$

holds, where $\|P\| = \sup_{\|x\|_X=1} |P(x)|$. We also study a variant of the problem in which we require the polynomials to be homogeneous.

Recall that a function $P : X \to \mathbb{K}$ is a continuous k-homogeneous polynomial if there is a continuous k-linear function $T : X^k \to \mathbb{K}$ for which $P(x) = T(x, \cdots, x)$. A function $Q : X \to \mathbb{K}$ is a continuous polynomial of degree k if $Q = \sum_{l=0}^{k} Q_l$ with Q_0 a constant, Q_l $(1 \leq l \leq k)$ an l-homogeneous polynomial and $Q_k \neq 0$.

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The factor problem has been studied by several authors. In [3], C. Benítez, Y. Sarantopoulos and A. Tonge proved that, for continuous polynomials, the inequality (1) holds with constant

$$M = \frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}$$

for any complex Banach space. The authors also showed that this is the best universal constant, since there are polynomials on ℓ_1 for which equality prevails. For complex Hilbert spaces and homogeneous polynomials, D. Pinasco proved in [10] that the optimal constant is

$$M = \sqrt{\frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}}.$$

This is a generalization of the result for linear functions obtained by Arias-de-Reyna in [1]. In [4], also for homogeneous polynomials, D. Carando, D. Pinasco and the author proved that for any complex $L_p(\mu)$ space, with $dim(L_p(\mu)) \geq n$ and $1 < p < 2$, the optimal constant is

$$M = \sqrt[p]{\frac{(k_1 + \cdots + k_n)^{(k_1+\cdots+k_n)}}{k_1^{k_1} \cdots k_n^{k_n}}}.$$

This article is partially motivated by the work of M. Lindström and R.A. Ryan in [8]. In that article they studied, among other things, a problem similar to (1): finding the so called polarization constant of a Banach space. They found a relation between the polarization constant of the ultraproduct $(X_i)_{\mathfrak{U}}$ and the polarization constant of each of the spaces X_i. Our objective is to do an analogous analysis for our problem (1). That is, to find a relation between the factor problem for the space $(X_i)_{\mathfrak{U}}$ and the factor problem for the spaces X_i.

In Section 2 we give some basic definitions and results of ultraproducts needed for our discussion. In Section 3 we state and prove the main result of this paper, involving ultraproducts, and a similar result on biduals.

2. Ultraproducts

We begin with some definitions, notations and basic results on filters, ultrafilters and ultraproducts. Most of the content presented in this section, as well as an exhaustive exposition on ultraproducts, can be found in Heinrich's article [7].

A filter $\mathfrak{U}$ on a family I is a collection of non-empty subsets of I closed by finite intersections and inclusions. An ultrafilter is maximal filter.

In order to define the ultraproduct of Banach spaces, we are going to consider some topological results first.

Definition 2.1. Let $\mathfrak{U}$ be an ultrafilter on I and X a topological space. We say that the limit of $(x_i)_{i\in I} \subseteq X$ with respect of $\mathfrak{U}$ is x if for every open neighborhood U of x the set $\{i \in I : x_i \in U\}$ is an element of $\mathfrak{U}$. We denote

$$\lim_{i,\mathfrak{U}} x_i = x.$$

The following is Proposition 1.5 from [7].

Proposition 2.2. *Let $\mathfrak{U}$ be an ultrafilter on I, X a compact Hausdorff space and $(x_i)_{i\in I} \subseteq X$. Then, the limit of $(x_i)_{i\in I}$ with respect of $\mathfrak{U}$ exists and is unique.*