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Deforming convex curves with fixed elastic energy ☆

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ABSTRACT

This paper presents a curve flow which preserves the elastic energy of the evolving curve. If the initial curve is a planar, simple and smooth curve with positive curvature then the local and global existence of the flow is proved. Under this flow, the evolving curve will converge to a finite circle in the C^∞ metric as time goes to infinity.

1. Introduction

In many fields, curve evolution problems are proposed to smooth curves from noisy input data, such as image processing [3], phase transitions [12], reverse engineering, image registration and mesh optimization [14], etc. Because of these various applications, the curve flows have been studied extensively. Many of the curve evolutions focus on the famous curve-shortening flow and its generalizations (see [5,8,9,1,7,17,4], etc.). If one needs to smooth the input data with some global quality preserved then he demands non-local flows, such as the area-preserving flows [6,18,20] and the length-preserving flows [19,22], etc. Since the elastic energy of a closed curve is a kind of interesting geometric quality that arouses some research interest (see [13,23,16]), we pose a flow in this paper to deform a given curve with its elastic energy preserved in order to obtain some more geometric properties of convex curves, such as geometric inequalities involving elastic energy. As far as we know, to construct such a kind of flow is the first try in this field.

Given a simple, closed and C^2 curve $X(\cdot)$ in the plane, one can define a kind of energy in the following form

$$E_n \triangleq \int_0^L (\kappa(s))^n ds,$$

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where n is a positive integer and κ is the relative curvature of the curve. If $n = 1$ then $E_1 = 2\pi$ (see Theorem 7 on page 20 of [24]). In the case of $n = 2$, E_2 is the so called elastic energy (or bending energy) of the curve (see the definition on page 1 of [23]).

Let $X_0 : S^1 \to \mathbb{R}^2$ be a smooth curve in the plane. We say X_0 is convex if it is a simple curve with positive curvature. Denote by $T(\varphi)$ the unit tangential vector of the curve at $X(\varphi)$. For each φ, let $N(\varphi)$ be the unit normal such that $\{T(\varphi), N(\varphi)\}$ gives a positive orientation of the plane $\mathbb{R}^2$. The support function of X_0 is defined by $p = -\langle X_0, N\rangle$. In this paper, the following evolution problem for convex curves will be investigated:

$$\begin{cases} \frac{\partial X}{\partial t}(\varphi, t) = \left(p(\varphi, t) - \frac{\int_0^{L(t)} \kappa^2(s,t)ds}{\int_0^{L(t)} \kappa^3(s,t)ds}\right) N(\varphi, t), \\ \qquad\qquad (\varphi, t) \in S^1 \times [0, \omega), \\ X(\varphi, 0) = X_0(\varphi), \quad \varphi \in S^1, \end{cases} \tag{1.1}$$

where $X(\varphi, t)$ is the evolving curve with its curvature, perimeter and support function denoted by $\kappa(\varphi, t)$, $L(t)$ and $p(\varphi, t)$, respectively. Under the flow (1.1), the elastic energy of the evolving curve is invariable, which makes this flow differ from all the previous ones.

Let θ be the oriented angle between the positive x-axis and the unit tangential vector of the curve. By the definition of the curvature κ, one has

$$\frac{d\theta}{ds} = \kappa.$$

So we can use $\theta \in [0, 2\pi]$ as a parameter for convex curves. From now on, we choose a convex curve to be the initial curve X_0 of our flow (1.1).

If the flow (1.1) has a solution on $S^1 \times [0, \omega)$ then the curvature of the evolving curve satisfies the following Cauchy problem (see Eq. (1.15) on page 20 of [4]):

$$\begin{cases} \frac{\partial \kappa}{\partial t}(\theta, t) = \kappa(\theta, t) - \frac{\int_0^{2\pi} \kappa(\theta,t)d\theta}{\int_0^{2\pi} \kappa^2(\theta,t)d\theta}\kappa^2(\theta, t), \quad (\theta, t) \in [0, 2\pi] \times [0, \omega), \\ \kappa(\theta, 0) = \kappa_0(\theta), \quad \theta \in [0, 2\pi], \end{cases} \tag{1.2}$$

where $\kappa_0(\theta)$ is the curvature of a convex curve in the plane and it satisfies the closing condition:

$$\int_0^{2\pi} \frac{e^{i\theta}}{\kappa_0(\theta)} d\theta = 0. \tag{1.3}$$

Although the support function p appears in the evolution equation (1.1), the shape of the evolving curve $X(\cdot, t)$ does not rely on the choice of the original point of the plane because $X(\cdot, t)$ is in fact uniquely determined by its curvature, the unique solution of (1.2)–(1.3).

In comparison with the previous work, the difficulty to investigate the flow (1.1) is first to prove the existence of the positive and uniformly bounded solution for the integro-differential equation (1.2)–(1.3) on time interval $[0, +\infty)$. And then we also need to study the asymptotic behavior of κ as time goes to infinity. To settle the first problem, we reduce the existence problem to find a fixed point in a closed set of a Banach space and this goal can be efficiently achieved by using Banach's fixed point theorem. Then we use comparison principles to prove the positivity and boundedness of κ in any finite time interval. In order to give a uniform bound of κ, we introduce a new trick to establish its Hanarck estimate. To deal with the second problem, we use some famous or important geometric inequalities, such as Bonnesen's inequality ([2] and [21]), Gage's inequality [5] and Lin–Tai inequality [15] to obtain the convergence of our flow. Our main theorem of the flow (1.1) is given as follows.