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Symmetry results of positive solutions of integral equations involving Riesz potential in exterior domains and in annular domains

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ABSTRACT

The purpose of this paper is to investigate positive solutions of integral equations involving Riesz potential. Exploiting the moving plane method in integral form, we give the radial symmetry of both the domain and solutions of our integral equations in exterior domains and in annular domains.

1. Introduction

The purpose of this paper is to study radial symmetry for positive solutions of Riesz potential type integral equations, which has attracted a lot of attention in the past decades. This symmetry problem of PDEs (in R^n and bounded domains) was investigated by Alexandroff [1] and Serrin [22] via the maximum principle and the moving plane method. It was further developed by Gidas, Ni and Nirenberg [8], Caffarelli, Gidas and Spruck [2], et al. As for the symmetry problems in exterior domains and in annular domains, they were studied by many authors, such as, Reichel [20], Moroz, Schaftingen [19], Y. Li [14], Gladiali, Grossi, Pacella and Srikanth [9] and so on.

Chen, Li and Ou first investigated in [4] the symmetry of positive solutions of Riesz potential integral equations in R^n by the Hardy–Littlewood–Sobolev inequality. Li, Ströhmer and Wang proved in [15] the symmetry of both the domain and positive solutions of Riesz potential equations in bounded domain Ω. One can refer to [11,12,16,18] and their references therein for further development. It is natural here to present whether potential integral equations have symmetry positive solutions in exterior domains and annular

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domains or not. In this paper, we answer this question and give some symmetry results. To the best of the author's knowledge, it is first time to study the symmetry of integral equations in exterior domains and in annular domains.

Assume Ω_1 is an open and bounded domain with $\partial\Omega_1 \in C^1$. We first study the following Riesz potential integral equations on the exterior domain $\Omega = R^n/\bar{\Omega}_1$.

$$\begin{cases} u(x) = \int\limits_{R^n} \frac{u^p(y)}{|x-y|^{n-\alpha}} dy, & x \in \Omega, \\ u(x) = C_1, & x \in R^n/\Omega, \end{cases} \tag{1.1}$$

with constants $0 < \alpha < n$, $p > 1$ and $C_1 > 0$.

We prove that

Theorem 1.1. *Assume $u \in L^q(\Omega)$ is a positive solution of Eq. (1.1) where $q = \frac{n(p-1)}{\alpha}$, $p > 1$, $0 < \alpha < n$. Then $\Omega_1 = R^n/\Omega$ must be a ball, u is radially symmetric and strictly decreasing with respect to the distance from the center of the ball.*

Remark 1.2. i) The condition $p > 1$ can be extended to $0 < p < +\infty$ with some additional regularity conditions. One can see [15] for detail ($u \in L^1(\Omega)$ provided $0 < p < 1$). To simplify the proof, we assume $p > 1$ here.

ii) If Ω_1 is empty, problem (1.1) becomes the problem in R^n (one can see the result in [4]) and our theorem is still valid.

iii) Heuristically, (1.1) is closely related to the following fractional differential equation

$$(-\Delta)^{\alpha/2} u = u^p, \quad x \in \Omega,$$

in the sense of distribution. P. Felmer and Y. Wang [6] have proved the symmetry of positive solutions to fractional equations in ball B_1 and in R^n respectively. S. Jarohs and T. Weth [13] studied this problem in bounded and symmetric domains. One can refer to [5] and [3] for more details.

The following general result holds by a similar proof of Theorem 1.1.

Corollary 1.3. *Suppose $p > 1$, $0 < \alpha < n$ and $q = \frac{np}{n-p(1+\alpha)} > 1$, Ω_1 is an open and bounded domain with $\partial\Omega_1 \in C^1$, $\Omega = R^n/\Omega_1$. Assume function $f(u)$ satisfies that*

$$\begin{cases} f(t) \geq 0, \ f'(t) \in L^p(\Omega), \\ f(t) \textit{ and } f'(t) \textit{ is nondecreasing with respect to } t. \end{cases}$$

If $u \in L^q(\Omega)$ is a positive solution of equation

$$\begin{cases} u(x) = \int\limits_{R^n} \frac{f(u(y))}{|x-y|^{n-\alpha}} dy, & x \in \Omega, \\ u(x) = C_1, & x \in R^n/\Omega. \end{cases}$$

Then Ω_1 must be a ball, u is radially symmetric and strictly decreasing with respect to the distance from the center of the ball.

As the second part of this paper, we investigate Riesz potential integral equations in an annular domain. Assume $\Omega_1 \subset \Omega_2 \subset R^n$ are bounded open domains with $\partial\Omega_1, \partial\Omega_2 \in C^1$ and $\partial\Omega_1 \cap \partial\Omega_2 = \varnothing$. We study a boundary value problem as follows.