J. Math. Anal. Appl. 427 (2015) 899–904

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Facial topology and extreme operators

Ana M. Cabrera-Serrano [1], Juan F. Mena-Jurado [*,2]

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071, Granada, Spain

ARTICLE INFO

Article history:
Received 17 July 2014
Available online 5 March 2015
Submitted by J. Bastero

Keywords:
Extreme operator
Continuous affine functions
Simplex
Facial topology

ABSTRACT

Let K be a simplex and let $\mathcal{A}_0(K)$ denote the space of continuous affine functions on K vanishing at a fixed extreme point, denoted by 0. We prove that if any extreme operator T from a Banach space X to $\mathcal{A}_0(K)$ is a nice operator (that is, T^*, the adjoint of T, preserves extreme points), then the facial topology of the set of extreme points different from 0 is discrete, and so $\mathcal{A}_0(K)$ is isometrically isomorphic to $c_0(I)$ for some set I. From here we derive the corresponding result for $\mathcal{A}(K)$, namely, if K is a simplex such that each extreme operator from any Banach space to $\mathcal{A}(K)$ is a nice operator, then the set of extreme points of K is finite.

1. Introduction

Given a real Banach space X, the Krein–Milman theorem asserts that the closed unit ball of the dual space, X^*, is the weak*-closed convex hull of its extreme points. Any extreme point of the unit ball of X^* is an operator from X into $\mathbb{R}$ whose adjoint operator takes extreme points of $\mathbb{R}$ into extreme points of X^*. Let Y be another real Banach space, then, taking into account the Banach–Alaoglu and Krein–Milman theorems, it is easy to prove that any operator from X into Y whose adjoint takes extreme points of the unit ball of Y^* into extreme points of the unit ball of X^* is an extreme point of the unit ball of the space of all operators from X into Y with its usual norm. An operator satisfying the above condition is called a *nice operator*. Nice operators have been studied in several papers (see, for example [2,6,10,11,14,15,17]). The term “nice operator” was first coined in [16]. However this notion was implicit in [4] in the context of spaces of continuous functions (i.e., spaces $\mathcal{C}(K)$, where K is a compact Hausdorff space), where nice operators are nothing else than weighted composition operators. In this last reference, the authors posed the problem about the coincidence of nice and extreme operators defined between spaces of continuous

* Corresponding author.
E-mail addresses: anich7@correo.ugr.es (A.M. Cabrera-Serrano), jfmena@ugr.es (J.F. Mena-Jurado).

1 Supported by Junta de Andalucía and FEDER Grant FQM-185.
2 Supported by Spanish MICINN and FEDER project No. MTM2012-31755 and by Junta de Andalucía and FEDER Grants FQM-185 and FQM-3737.

http://dx.doi.org/10.1016/j.jmaa.2015.03.003
0022-247X/

functions. Several positive results have been obtained since then (see [4,9,16,18,19,22]), but Sharir proved in [20,21] the existence of non-nice extreme operators in $\mathcal{C}(K)$-type spaces. Sharir also proved that nice and extreme operators between L_1-spaces always agree [19, Theorem 2.2].

Now we introduce some notation which is usual in Banach spaces. All Banach spaces are assumed to be real. For a Banach space X, B_X and E_X will stand for the closed unit ball of X and the set of extreme points of B_X, respectively. If Y is another Banach space, $\mathcal{L}(X,Y)$ will denote the space of all operators (linear and continuous mappings) from X into Y provided with the operator norm. For each T in $\mathcal{L}(X,Y)$, the adjoint operator of T is denoted by T^*. The dual space of X is $X^* = \mathcal{L}(X,\mathbb{R})$.

The spaces appearing in the following definition were introduced and studied for the first time in [5]. These spaces share with $\mathbb{R}$ the aforementioned property that extreme operators from any Banach space into them are nice operators.

Definition 1.1. A Banach space X is said to be "*nice*" if for any Banach space Y every extreme operator in the unit ball of $\mathcal{L}(Y,X)$ is a nice operator, that is, $T^*(E_{X^*}) \subseteq E_{Y^*}$.

In [5], nice Banach spaces were characterized in the context of several classical Banach spaces (see, for example, [5, Corollaries 2.5 and 2.6]). Finite-dimensional nice Banach spaces were also described [5, Theorem 2.12].

The main goal in this paper is to study spaces of (real) continuous affine functions defined on compact convex sets which are nice. We refer to [1] for the notions which we introduce below.

Let K be compact convex subset of some (real) locally convex Hausdorff space and let ∂K stand for the set of extreme points of K. For $F \subseteq K$, the *complementary set* F' is the union of all faces of K disjoint from F. A face F of K is said to be a *split face* if F' is convex and every point in $K \setminus (F \cup F')$ can be uniquely represented as a convex combination of a point in F and a point in F'. It can be easily proved that F' is a split face whenever F is also a split face. The sets $F \cap \partial K$, where F is a closed split face of K, are the collection of closed sets for a topology in ∂K, which is called the *facial topology* of ∂K. If one considers a square in the plane, then the only split faces are trivial and so the facial topology is the trivial topology. This example shows that the facial topology is non-Hausdorff in general although ∂K is always facially compact [1, Proposition II.6.21].

Now we take advantage of the split face notion to introduce the concept of simplex. A compact convex set K is said to be a *simplex* if every closed face of K is a split face. Indeed this is a characterization of the usual definition of simplex proved in [8]. Therefore, if K is a simplex, every point in ∂K is facially closed.

We will denote by $\mathcal{A}(K)$ the space of all (real) continuous affine functions on K with the usual supremum norm and $\mathcal{A}_0(K)$ will be the subspace of $\mathcal{A}(K)$ of all functions vanishing at a fixed point in ∂K.

The main result in this paper is the characterization of spaces $\mathcal{A}_0(K)$ which are nice when K is a simplex. As a result we characterize nice spaces of type $\mathcal{A}(K)$ with K a simplex. As a consequence we also get the description of nice spaces of type $\mathcal{C}_0(L)$. This last result was obtained in [5, Corollary 2.5].

2. The results

One of the main tools in what follows will be the following theorem proved in [5].

Theorem 2.1. *(See [5, Theorem 2.2].) Let X be a Banach space such that there exists $e_0^* \in E_{X^*}$ which satisfies:*

i) $X^* = \overline{\operatorname{lin}(E_{X^*} \setminus \{\pm e_0^*\})}^{w^*}$.
ii) *For each $e^* \in E_{X^*} \setminus \{\pm e_0^*\}$, there exists $x \in B_X$ such that $e_0^*(x) = 0$ and $e^*(x) = 1$.*

Then X is non-nice.