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# Singular perturbations involving fast diffusion

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A B S T R A C T

We study a couple of examples of pure and applied mathematics where interesting singular limits are obtained when diffusion's speed increases to infinity. These include the recent models of kinase activity [17,29] and of early lung-cancer [37,38], and the example of Khasminskii [32].



## 1. Introduction

One of the fundamental properties of diffusion in a bounded domain is that it 'averages' solutions (of the heat equation) over the domain (see e.g. [47, Theorem 14.17]). The effect is probably best known in the context of Neumann boundary conditions: To restrict ourselves to the simplest one-dimensional case, let $C[a,b]$ be the space of continuous functions on an interval $[a,b]$ $(a < b)$ and let $\triangle$ be the Laplacian: $\triangle f = f''$ with domain

$$D(\triangle) = \{f \in C^2[a,b]; f'(a) = f'(b) = 0\},$$

where $C^2[a,b]$ is the subspace of twice continuously differentiable members of $C[a,b]$. Then, denoting by $\left(\mathrm{e}^{t\triangle}\right)_{t\geq 0}$ the semigroup generated by $\triangle$, we have

$$\lim_{t\to\infty} \mathrm{e}^{t\triangle} f = \frac{1}{b-a}\int_a^b f, \tag{1}$$

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where the number on the right-hand side is identified with a constant function on $[a, b]$. (A physical interpretation of this result is that as time passes the temperature distribution in an isolated finite rod 'averages out' and becomes constant throughout the rod.)

Similarly, let $C_{\mathrm{p}}[a, b]$ be the subspace of periodic functions $f \in C[a, b]$, i.e., functions such that $f(a) = f(b)$. Then, the Laplacian with domain

$$D(\triangle) = \{f \in C^2[a, b] \cap C_{\mathrm{p}}[a, b]; f'' \in C_{\mathrm{p}}[a, b]\}$$

is the generator of a semigroup in $C_{\mathrm{p}}[a, b]$ and (1) holds in this case also.

Our third and final example involves $C_0(0, 1]$, the subspace of $f \in C[0, 1]$ with $f(0) = 0$. Here (see Section 3), the Laplacian with domain

$$D(\triangle) = \{f \in C^2[0, 1]; f(0) = f''(0) = 0 \text{ and } f'(1) = f(1)\}$$

generates a semigroup, too, and we have

$$\lim_{t\to\infty} \mathrm{e}^{t\triangle} f = 3 \int_0^1 x f(x) \, \mathrm{d}x \, h, \tag{2}$$

where $h(x) = x$, $x \in (0, 1]$.

The homogenization effect of diffusion described above, when coupled with other physical or biological forces leads to intriguing singular perturbations. For example, in [10,11] it was shown that in studying fast diffusions on a graph with semipermeable membranes at vertices, provided transmission rates through the membranes are small, one may approximate the process involved by a Markov chain on the state-space composed of edges 'lumped' into single points (forming vertices of the 'dual' line-graph [19]). Moreover, in the Alt and Lauffenburger [3] model of leukocytes reacting to a bacterial invasion by moving up a gradient of some chemical attractant produced by the bacteria (see Section 13.4.2 in [31]) a system of three PDEs is reduced to one equation provided bacterial diffusion is much smaller than the diffusion of leukocytes or of chemoattractants (which is typically the case).

In this paper we exhibit three examples of such singular perturbations. After presenting semigroup-theoretical tools in Section 2, we study the fast diffusion limit in the recent model of kinase activity [29] in Section 3 (this limit involves (2)). In Section 4, we analyze the early carcinogenesis model of Marciniak-Czochra and Kimmel [36–38] (involving Neumann boundary conditions). The final Section 5 of the paper is devoted to the semigroup-theoretical treatment of Khasminskii's classical example [32], where periodic boundary conditions are used.

## 2. Semigroup-theoretical tools

### *2.1. Convergence of semigroups*

In proving convergence theorems for semigroups, one usually resorts to the Trotter–Kato theorem or its Sova–Kurtz version [5,21,22,25,42]. However, these theorems give necessary and sufficient conditions for convergence that is almost uniform in $t \geq 0$, and cannot be applied to singular perturbations, where the convergence is typically almost uniform merely in $t > 0$ (see e.g. [8] and references given there). Luckily, a surprising number of examples of singular perturbations fall into the following scheme, devised by T.G. Kurtz [22, pp. 39–42], [34,35].

Let $(\epsilon_n)_{n\geq 1}$ be a sequence of positive numbers converging to 0. Suppose $A_n$, $n \geq 1$ are generators of equi-bounded semigroups $\{\mathrm{e}^{tA_n}, t \geq 0\}$ in a Banach space $\mathbb{X}$, and $Q$ generates a strongly continuous semigroup $\left(\mathrm{e}^{tQ}\right)_{t\geq 0}$ such that