J. Math. Anal. Appl. 427 (2015) 1084–1100

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the topological entropy of a semigroup of continuous maps

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A R T I C L E I N F O

Article history:
Received 20 June 2014
Available online 3 March 2015
Submitted by B. Bongiorno

Keywords:
Topological entropy
Semigroup of continuous maps
Affine transformations

A B S T R A C T

We introduce the notion of the lower local entropy of a Borel probability measure on a compact metric space to estimate the bounds of topological entropy with respect to any subset in the case of finitely generated semigroup of continuous maps. Moreover, we give the notion of the topological entropy of a semigroup generated by finite uniformly continuous maps on a metric space not necessarily compact, provide some properties of this topological entropy, and estimate the bounds of them for some particular systems, such as a semigroup generated by finite affine transformations on the p-dimensional torus and a semigroup generated by finite smooth maps on Riemannian manifolds.

1. Introduction

Let T be a continuous map acting on a compact topological space X. The notion of topological entropy of T, which was introduced by Adler, Konheim and McAndrew [1] as an invariant of topological conjugacy, describes the complexity of a system. Later, Bowen [4] and Dinaburg [7] gave equivalent approaches to the notion of entropy in the case X is a metrizable space. Bowen [5], Pesin and Pitskel [21] gave characterizations of dimension type for topological entropy and topological pressure. Since the entropy appeared to be a very useful invariant in ergodic theory and dynamical systems, there were several attempts to find its suitable generalizations for other systems such as groups, pseudogroups, graphs, foliations, nonautonomous dynamical systems and so on [8,11–13,17–19,23]. Ghys, Langevin, and Walczak in [9] proposed a definition of a topological entropy for finitely generated pseudogroups of continuous transformations. Biś [2] introduced the notion of topological entropy of a semigroup of finite continuous maps. Moreover, Biś and Urbański in [3] investigate the notion of topological entropy of a semigroup of continuous maps and provide several of its basic properties. Later, for the case of finitely generated semigroup of continuous maps, Ma and

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http://dx.doi.org/10.1016/j.jmaa.2015.02.082
0022-247X/

Wu [16], Ma and Liu [14] introduced the notions of topological pressure and topological entropy by using the Carathéodory–Pesin structure (C–P structure) defined in [20] and gave some properties of them.

In this paper, to estimate the bounds of topological entropy with respect to any subset in the case of finitely generated semigroup of continuous maps, we introduce the notion of the lower local entropy of a Borel probability measure on a compact metric space. Moreover, we extend the notion of the topological entropy of a semigroup generated by finite continuous maps on a compact metric space to the topological entropy of a semigroup generated by finite uniformly continuous maps on a metric space not necessarily compact, and estimate the bounds of the extended topological entropy for some particular systems, such as a semigroup generated by finite affine transformations on metrizable groups and a semigroup generated by finite smooth maps on Riemannian manifolds.

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we give the definitions of the lower local entropy of a Borel probability measure on a compact metric space and the topological entropy of a semigroup generated by finite uniformly continuous maps on a metric space not necessarily compact, and give some fundamental properties of them which are useful to calculate them. In Section 4, we estimate the bounds of topological entropy with respect to any subset by using the lower local entropy of a Borel probability measure, and the bounds of the extended topological entropy for some particular systems, such as a semigroup generated by finite affine transformations on the p-dimensional torus and a semigroup generated by finite smooth maps on Riemannian manifold.

2. Preliminaries

Let X be a compact metric space with metric d. Consider a semigroup G of continuous transformations of X into itself. The semigroup G is assumed to be finitely generated, e.g. there exists a finite set $G_1 = \{f_1, f_2, \cdots, f_k\}$ such that $G = \bigcup_{n\in\mathbb{N}} G_n$, where $G_n = \{g_1 \circ \cdots \circ g_n : g_1, \cdots, g_n \in G_1\}$. We always assume that $f_1 = id_X$, the identity map on X is in G_1. This implies that $G_m \subset G_n$ for all $m \le n$. Define a new metric d_n on X by $d_n(x, y) = \max\{d(g(x), g(y)) : g \in G_n\}$. For any integer $n \ge 1$, real number $r > 0$ and $x \in X$, define the (n, r)-ball centered at x by

$$B_n(x, r) = \{y \in X : d_n(x, y) < r\}.$$

A subset $F \subseteq X$ is said to be an (n, ϵ)-separated subset of X, if $x, y \in F$, $x \ne y$ implies $d_n(x, y) > \epsilon$. Define $s(n, \epsilon, X, G_1)$ to be the largest cardinality of any (n, ϵ)-separated set of X. Since X is compact with respect to the metric d, $s(n, \epsilon, X, G_1)$ is a finite number.

In [2], Biś defined

$$h(G_1) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log(s(n, \epsilon, X, G_1)).$$

The quantity $h(G_1)$ is called the topological entropy of a semigroup G generated by G_1.

We also describe $h(G_1)$ in terms of (n, ϵ)-spanning sets. A subset E of X is called (n, ϵ)-spanning set of X if for every $x \in X$ there exists $y \in E$ such that $d_n(x, y) \le \epsilon$. Let $r(n, \epsilon, X, G_1)$ denote the smallest cardinality of any (n, ϵ)-spanning sets of X. Biś [2] showed that for any semigroup G generated by a finite set G_1 the following equality holds

$$h(G_1) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log(r(n.\epsilon, X, G_1)).$$

It is easy to see that the topological entropy of a semigroup is a generalization of the topological entropy of a continuous map. Indeed, let $f : X \to X$ be a continuous transformation of a compact space X and $G(f)$