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Conditional Lie–Bäcklund symmetry, second-order differential constraint and direct reduction of diffusion systems

Jianping Wang [a], Lina Ji [a,b,*]

[a] Department of Information & Computational Science, Henan Agricultural University, Zhengzhou, 450002, PR China

[b] Department of Mathematics, Shanghai University, Shanghai, 200444, PR China

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ABSTRACT

The equivalence relation between conditional Lie–Bäcklund symmetry, higher-order differential constraint and direct reduction is presented in the case of evolution systems. The second-order nonlinear conditional Lie–Bäcklund symmetries and corresponding induced differential constraints admitted by nonlinear diffusion systems are identified as an application of the conditional Lie–Bäcklund symmetry of evolution systems. Consequently, the corresponding symmetry reductions are obtained due to the compatibility of the resulting differential constraints and the governing systems.

1. Introduction

The classical symmetry, first introduced by Sophus Lie in [32], unifies systematically various specialized solution methods for ordinary differential equations (ODEs) including integrating factor, reduction of order, undetermined coefficient and Laplace transform. The theories and applications of Lie's classical symmetry for various classes of partial differential equations (PDEs) are discussed in a series of monographs by Ovsiannikov [39], Bluman and Kumei [4], Olver [35], Ibragimov [23], among others.

Over the years, many generalizations of the concept of symmetry groups of nonlinear PDEs have been proposed. Bluman and Cole [3] suggested the so-called nonclassical method. Clarkson and Kruskal [15] created the direct method, which is equivalent to Bluman and Cole's nonclassical approach if certain criterion is satisfied [2]. Fushchych et al. [17] proposed conditional symmetry, which is a generalization of the nonclassical symmetry. Olver and Roaenau [37,38] generalized the nonclassical method to the weak symmetry method and the side condition method. Ovsiannikov [39] developed the method of partially invariant solutions. Fokas and Liu [16] carried out the conditional Lie–Bäcklund symmetry (CLBS) for the

* Corresponding author at: Department of Information & Computational Science, Henan Agricultural University, Zhengzhou, 450002, PR China.
E-mail address: jilina@henau.edu.cn (L. Ji).

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special case when evolution equations and symmetries do not explicitly involve time variable. Zhdanov [49] considered CLBS of evolution equation for the general case. There are a lot of symmetry related methods such as the nonlinear separation method [18], the sign-invariant method [20,21] and the invariant subspace method [19,22], which are all included within the framework of CLBS [24,25,40,41,44].

Practically all these methods for finding exact particular solutions of PDEs require the analysis of the compatibility of over-determined system. It is concluded in [37,38] that "the unifying theme below finding special solutions of PDEs is not, as is commonly supposed, group theory, but rather the more analytic subject of over-determined system of the considered PDEs". Olver [36–38], Kaptsov [1], Levi and Winternitz [31] show that many reduction methods such as nonclassical symmetry, partial invariance, separation of variables and direct method can be reformulated by using the technicalities of the method of differential constraints.

Differential constraints arose originally in the theory of PDEs of the first-order. The theory of differential constraints has its origins in the work of Yanenko [48] on gas dynamic. A survey of this method was presented by Sidorov, Shapeev and Yanenko in [46]. The general formulation of the method of differential constraints requires that the original system of PDEs

$$F^1 = 0, \ F^2 = 0, \ \cdots, \ F^m = 0 \tag{1.1}$$

is enlarged by appending additional differential equations (differential constraints)

$$h_1 = 0, \ h_2 = 0, \ \cdots, \ h_p = 0 \tag{1.2}$$

such that the over-determined system (1.1), (1.2) satisfies some conditions of compatibility.

However, the problem of finding all differential constraints compatible with certain equations can be more complicated than the investigation of the original equations. Therefore it is better to content oneself with finding constraints in some classes, and these classes must be chosen using additional considerations. The most popular way is to append to (1.1) a system of first-order differential equations defined by the invariant surface conditions associated with a group as a symmetry group, which yields that the first-order differential constraint is exactly equivalent to nonclassical symmetry. The differential constraint is the invariant surface condition and its compatibility with the PDE implies that the ansatz based on similarity variables (group invariants) reduces the equation to a system of ODEs.

Olver [36] concluded that Galaktionov's nonlinear separation solution is admitted by a PDE if and only if a second-order ODE is compatible with the considered equation. On the other hand, a number of such solutions are constructed due to the compatibility of the second-order ODE induced by the characteristic of CLBS and the considered equations in [40,41]. For scalar PDE, the second-order differential constraint is equivalent to second-order CLBS and so does for the higher-order case [30].

Kaptsov [27] proposed the linear determining equation for finding differential constraints of PDEs which are more general than the classical determining equations for Lie generators. Second-order and third-order nonlinear differential constraints of nonlinear diffusion equations have been constructed due to the method of linear determining equations in [28]. Cherniha [6] suggested additional generating conditions in the form of a linear higher-order ordinary differential equation to construct exact solutions of nonlinear evolution equations, which is very effective to study reductions of scalar diffusion equations and diffusion systems [6,7,13].

In this paper, we will establish one-to-one correspondence between CLBS and differential constraint for evolution systems. The studies of CLBS and differential constraint for evolution systems are much less than what for scalar evolution equations. The linear CLBSs with time independent coefficients of diffusion systems are discussed in [42,51], where this problem is restated in terms of the so-called invariant subspace method. The nonlinear CLBS related to invariant subspace is used to classify and seek for symmetry reductions of diffusion systems in [26], where the nonlinear CLBS can be linearized due to certain transformation. We will study other type of nonlinear CLBS for the two-component nonlinear diffusion system