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Weak sequential completeness of spaces of homogeneous polynomials

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ABSTRACT

Let $\mathcal{P}_w(^nE;F)$ be the space of all continuous n-homogeneous polynomials from a Banach space E into another F, that are weakly continuous on bounded sets. We give sufficient conditions for the weak sequential completeness of $\mathcal{P}_w(^nE;F)$. These sufficient conditions are also necessary if both E^* and F have the bounded compact approximation property. We also show that the weak sequential completeness and the reflexivity of $\mathcal{P}_w(^nE;F)$ are equivalent whenever both E and F are reflexive.

1. Introduction

For Banach spaces E and F, let $\mathcal{P}(^nE;F)$ be the space of all continuous n-homogeneous polynomials from E into F. After the pioneer work of Ryan [26], several authors (e.g. see [1,2,19,24,25]) have searched for necessary and sufficient conditions for the reflexivity of $\mathcal{P}(^nE;F)$. Among them, Alencar [1] gave necessary and sufficient conditions for the reflexivity of $\mathcal{P}(^nE;\mathbb{C})$ under the hypothesis of the approximation property of E, and Mujica [24] gave necessary and sufficient conditions for the reflexivity of $\mathcal{P}(^nE;F)$ under the hypothesis of the compact approximation property of E.

A property closely related to the reflexivity is the weak sequential completeness. In Section 3 of this paper, we give sufficient conditions for the weak sequential completeness of $\mathcal{P}_w(^nE;F)$, the subspace of all P in $\mathcal{P}(^nE;F)$ that are weakly continuous on bounded sets. We show that these sufficient conditions are also necessary when both E^* and F have the bounded compact approximation property.

In Section 4, we show that the weak sequential completeness and the reflexivity of $\mathcal{P}_w(^nE;F)$ coincide whenever both E and F are reflexive. As a consequence, a result of Mujica [24] about the reflexivity of $\mathcal{P}(^nE;F)$ is obtained.

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Mujica [23] showed that the (bounded) approximation property is inherited by the symmetric projective tensor products. In Section 5, we show that the (bounded) compact approximation property is also inherited by the (symmetric) projective tensor products. However, we note that Aron and Schottenloher's counterexample [7] shows that the (bounded) compact approximation property is not inherited by the spaces of homogeneous polynomials in general.

2. Preliminaries

Throughout the paper, E and F are Banach spaces over the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Denote by $\mathcal{L}(E;F)$, $\mathcal{K}(E;F)$, and $\mathcal{W}(E;F)$, respectively, the spaces of all bounded, all compact, and all weakly compact linear operators from E into F. For a bounded linear operator $T : E \to F$, let $T[E]$ denote the image of T and let $T^* : F^* \to E^*$ denote the adjoint operator (i.e., the dual map) of T.

Let n be a positive integer. A map $P : E \to F$ is said to be a continuous n-homogeneous polynomial if there is a continuous symmetric n-linear map T from $E \times \cdots \times E$ (a product of n copies of E) into F such that $P(x) = T(x, \dots, x)$. Indeed, the symmetric n-linear operator $T_P : E \times \cdots \times E \to F$ associated to P can be given by the *Polarization Formula*:

$$T_P(x_1,\dots,x_n) = \frac{1}{2^n n!} \sum_{\epsilon_i=\pm 1} \epsilon_1 \cdots \epsilon_n P\Big(\sum_{i=1}^{n} \epsilon_i x_i\Big), \quad \forall x_1,\dots,x_n \in E.$$

Let $\mathcal{P}(^nE;F)$, $\mathcal{P}_w(^nE;F)$, and $\mathcal{P}_{wsc}(^nE;F)$, respectively, denote the space of all continuous n-homogeneous polynomials from E into F, the subspace of all P in $\mathcal{P}(^nE;F)$ that are weakly continuous on bounded sets, and the subspace of all P in $\mathcal{P}(^nE;F)$ that are weakly sequentially continuous. In particular, if $F = \mathbb{R}$ or $\mathbb{C}$, then $\mathcal{P}(^nE;F)$, $\mathcal{P}_w(^nE;F)$, and $\mathcal{P}_{wsc}(^nE;F)$ are simply denoted by $\mathcal{P}(^nE)$, $\mathcal{P}_w(^nE)$, and $\mathcal{P}_{wsc}(^nE)$, respectively. It is known that

$$\mathcal{P}_w(^nE;F) \subseteq \mathcal{P}_{wsc}(^nE;F) \subseteq \mathcal{P}(^nE;F) \tag{2.1}$$

and that $\mathcal{P}_w(^nE;F) = \mathcal{P}_{wsc}(^nE;F)$ for any $n \in \mathbb{N}$ if and only if E contains no copy of ℓ_1 (see [5, Prop. 2.12], also see [14, p. 116, Prop. 2.36]).

Let $\otimes_n E$ denote the *n-fold algebraic tensor product* of E. For $x_1 \otimes \cdots \otimes x_n \in \otimes_n E$, let $x_1 \otimes_s \cdots \otimes_s x_n$ denote its symmetrization, that is,

$$x_1 \otimes_s \cdots \otimes_s x_n = \frac{1}{n!} \sum_{\sigma \in \pi(n)} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)},$$

where $\pi(n)$ is the group of permutations of $\{1,\dots,n\}$. Let $\otimes_{n,s}E$ denote the *n-fold symmetric algebraic tensor product* of E, that is, the linear span of $\{x_1 \otimes_s \cdots \otimes_s x_n : x_1,\dots,x_n \in E\}$ in $\otimes_n E$. Let $\hat{\otimes}_{n,s,\pi}E$ denote the *n-fold symmetric projective tensor product* of E, that is, the completion of $\otimes_{n,s}E$ under the *symmetric projective tensor norm* on $\otimes_{n,s}E$ defined by

$$\|u\| = \inf\left\{\sum_{k=1}^{m} |\lambda_k| \cdot \|x_k\|^n : x_k \in E, u = \sum_{k=1}^{m} \lambda_k x_k \otimes \cdots \otimes x_k\right\}, \quad u \in \otimes_{n,s}E.$$

Define $\theta_n : E \to \hat{\otimes}_{n,s,\pi}E$ by $\theta_n(x) = x \otimes \cdots \otimes x$ for every $x \in E$. Then $\theta_n \in \mathcal{P}(^nE; \hat{\otimes}_{n,s,\pi}E)$. For every $P \in \mathcal{P}(^nE;F)$, let $A_P \in \mathcal{L}(\hat{\otimes}_{n,s,\pi}E;F)$ denote its linearization, that is, $P = A_P \circ \theta_n$. Then under the isometry: $P \to A_P$, the Banach space $\mathcal{P}(^nE;F)$ is isometrically isomorphic to $\mathcal{L}(\hat{\otimes}_{n,s,\pi}E;F)$. This implies that $\mathcal{P}(^nE) = \mathcal{P}_{wsc}(^nE)$ if and only if $\theta_n : E \to \hat{\otimes}_{n,s,\pi}E$ is sequentially continuous with respect to the weak topology of E and the weak topology of $\hat{\otimes}_{n,s,\pi}E$.