J. Math. Anal. Appl. 427 (2015) 1131–1167

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Local existence results for the Westervelt equation with nonlinear damping and Neumann as well as absorbing boundary conditions

Vanja Nikolić

Institut für Mathematik, Alpen-Adria-Universität Klagenfurt, Universitätsstraße 65-57, 9020 Klagenfurt am Wörthersee, Austria

ARTICLE INFO

Article history:
Received 19 August 2014
Available online 27 February 2015
Submitted by D.M. Ambrose

Keywords:
Nonlinear acoustics
Westervelt's equation
Local existence

ABSTRACT

We investigate the Westervelt equation with several versions of nonlinear damping and lower order damping terms and Neumann as well as absorbing boundary conditions. We prove local in time existence of weak solutions under the assumption that the initial and boundary data are sufficiently small. Additionally, we prove local well-posedness in the case of spatially varying L^∞ coefficients, a model relevant in high intensity focused ultrasound (HIFU) applications.

1. Introduction

High intensity focused ultrasound (HIFU) is crucial in many medical and industrial applications including lithotripsy, thermotherapy, ultrasound cleaning or welding and sonochemistry. Widely used mathematical model for nonlinear wave propagation is the Westervelt equation, which can either be written in terms of the acoustic pressure p

$$(1-2kp)p_{tt} - c^2\Delta p - b\Delta p_t = 2k(p_t)^2, \tag{1.1}$$

or in terms of the acoustic velocity potential ψ

$$(1-2\tilde{k}\psi_t)\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = 0, \tag{1.2}$$

with $\varrho\psi_t = p$. Here, c denotes the speed and b the diffusivity of sound, $k = \beta_a/\lambda$, $\beta_a = 1 + B/(2A)$, B/A represents the parameter of nonlinearity, ϱ is the mass density, $\lambda = \varrho c^2$ is the bulk modulus and $\tilde{k} = \varrho k$. For a detailed derivation of (1.1) and (1.2) we refer the reader to [4,9,13].

E-mail address: vanja.nikolic@aau.at.

http://dx.doi.org/10.1016/j.jmaa.2015.02.076

Well-posedness and exponential decay of small and H^2-spatially regular solutions is established for the Westervelt equation with homogeneous [6] and inhomogeneous [8] Dirichlet and Neumann [7] boundary conditions as well as with boundary instead of interior damping [5].

A significant task in the analysis of the Westervelt equation is avoiding degeneracy of the coefficient $1-2kp$ for the second time derivative p_{tt} in (1.1) and, similarly, of the term $1-2\tilde{k}\psi_t$ in the formulation (1.2). At the same time, in applications the existence of spatially less regular solutions is important, e.g. in the coupling of acoustic with acoustic or elastic regions with different material parameters. In [2], Brunnhuber, Kaltenbacher and Radu treated this issue by introducing nonlinear damping terms to the Westervelt equation and considering the following equations

$$(1-2ku)u_{tt} - c^2\Delta u - b\,\mathrm{div}\Big(((1-\delta)+\delta|\nabla u_t|^{q-1})\nabla u_t\Big) = 2k(u_t)^2, \tag{1.3}$$

$$(1-2ku)u_{tt} - c^2\mathrm{div}(\nabla u + \varepsilon|\nabla u|^{q-1}\nabla u) - b\Delta u_t = 2k(u_t)^2, \tag{1.4}$$

$$u_{tt} - \frac{c^2}{1-2\tilde{k}u_t}\Delta u - b\,\mathrm{div}\Big(((1-\delta)+\delta|\nabla u_t|^{q-1})\nabla u_t\Big) = 0, \tag{1.5}$$

with homogeneous Dirichlet boundary data. First two equations are derived from the Westervelt equation in the acoustic pressure formulation (1.1), while the third equation comes from the acoustic potential formulation (1.2) (with the notation changed to $p \to u$, $\psi \to u$). Added nonlinear damping terms make obtaining $L^\infty(0,T;L^\infty(\Omega))$ estimate on u (u_t) possible, without the need to estimate Δu (Δu_t) and thus refraining from too high regularity.

The central aim of the present paper is to investigate this relaxation of regularity by nonlinear damping, but equipped with practically relevant absorbing and Neumann boundary data. This is motivated by many applications of high intensity focused ultrasound where the need for more realistic boundary conditions is evident. E.g. in lithotripsy one faces the problem of a physically unbounded domain, as typical in acoustics, which should be truncated for numerical computations. Absorbing boundary conditions are then used to avoid reflections on the artificial boundary $\hat{\Gamma}$ of the computational domain.

Ultrasound excitation, e.g. by piezoelectric transducers, can be modeled by Neumann boundary conditions on the rest of the boundary $\Gamma = \partial\Omega \setminus \hat{\Gamma}$.

In our case, the design of the nonlinear absorbing and inhomogeneous Neumann boundary conditions is influenced by the presence of the nonlinear strong damping in the equations. We will study initial boundary value problems of the following type:

$$\begin{cases} (1-2ku)u_{tt} - c^2\Delta u - b\,\mathrm{div}\Big(((1-\delta)+\delta|\nabla u_t|^{q-1})\nabla u_t\Big) + \beta u_t = 2k(u_t)^2 \text{ in } \Omega\times(0,T], \\ c^2\frac{\partial u}{\partial n} + b((1-\delta)+\delta|\nabla u_t|^{q-1})\frac{\partial u_t}{\partial n} = g \text{ on } \Gamma\times(0,T], \\ \alpha u_t + c^2\frac{\partial u}{\partial n} + b((1-\delta)+\delta|\nabla u_t|^{q-1})\frac{\partial u_t}{\partial n} = 0 \text{ on } \hat{\Gamma}\times(0,T], \\ (u,u_t) = (u_0,u_1) \text{ on } \overline{\Omega}\times\{t=0\}, \end{cases} \tag{1.6}$$

$$\begin{cases} (1-2ku)u_{tt} - c^2\Delta u - b\,\mathrm{div}\Big(((1-\delta)+\delta|\nabla u_t|^{q-1})\nabla u_t\Big) + \gamma|u_t|^{q-1}u_t = 2k(u_t)^2 \text{ in } \Omega\times(0,T], \\ c^2\frac{\partial u}{\partial n} + b((1-\delta)+\delta|\nabla u_t|^{q-1})\frac{\partial u_t}{\partial n} = g \text{ on } \Gamma\times(0,T], \\ \alpha u_t + c^2\frac{\partial u}{\partial n} + b((1-\delta)+\delta|\nabla u_t|^{q-1})\frac{\partial u_t}{\partial n} = 0 \text{ on } \hat{\Gamma}\times(0,T], \\ (u,u_t) = (u_0,u_1) \text{ on } \overline{\Omega}\times\{t=0\}, \end{cases} \tag{1.7}$$

$$\begin{cases} (1-2ku)u_{tt} - c^2\mathrm{div}(\nabla u + \varepsilon|\nabla u|^{q-1}\nabla u) - b\Delta u_t + \beta u_t = 2k(u_t)^2 \text{ in } \Omega\times(0,T], \\ c^2\frac{\partial u}{\partial n} + c^2\varepsilon|\nabla u|^{q-1}\frac{\partial u}{\partial n} + b\frac{\partial u_t}{\partial n} = g \text{ on } \Gamma\times(0,T], \\ \alpha u_t + c^2\frac{\partial u}{\partial n} + c^2\varepsilon|\nabla u|^{q-1}\frac{\partial u}{\partial n} + b\frac{\partial u_t}{\partial n} = 0 \text{ on } \hat{\Gamma}\times(0,T], \\ (u,u_t) = (u_0,u_1) \text{ on } \overline{\Omega}\times\{t=0\}, \end{cases} \tag{1.8}$$