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Local approach to Kadec–Klee properties in symmetric function spaces ☆

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ABSTRACT

We prove several results concerning local approach to Kadec–Klee properties with respect to global (local) convergence in measure in symmetric Banach function spaces which may be of independent interest. Moreover, we prove characterizations of these properties in the Lorentz spaces. Finally, we show applications of H_g and H_l points to the local best dominated approximation problems in Banach lattices.

1. Introduction

Geometry of Banach spaces has been intensively developed during the last decades, since it has found a lot of applications in many branches of mathematics. The metric geometry deals with properties invariant under isometries (for example rotundity, uniform rotundity and many intermediate properties). The monotonicity properties (strict and uniform monotonicity) play an analogous role in the geometry of Banach lattices. However, the studies of global properties are not always sufficient. When the Banach space (Banach lattice) has not the global property then it is natural to ask about the local structure. This leads among others to the notion of an extreme point. The respective role in the theory of Banach lattices play the points of lower and upper monotonicity. The local geometry has been deeply investigated recently (see [7,14,16,26–28]) and one of the important reasons is an application to local best dominated approximation problems in Banach lattices (see [7]).

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In Section 2 we recall the necessary terminology.

Section 3 is devoted to symmetric Banach function spaces. The essential question in the global geometry is whether a geometric property can be equivalently considered only on the positive cone E_+ of E (see [21,22] for further references). We prove the local version of such result, namely a point x is an H_g point if and only if $|x|$ is an H_g point. The more delicate question is whether a point x has some local property P if and only if its nonincreasing rearrangement x^* has the same property P and the positive answer is very useful in verifying local properties in particular classes of symmetric function spaces (see [7]). The goal of this paper is to study the structure of H_g and H_l points from that point of view. Moreover, we will show the relationships between H_g, H_l points and points of upper monotonicity, generalizing the global characterization from [5]. Furthermore, we prove that, for an H_g point, the norm is lower semicontinuous with respect to the global convergence in measure, similarly as, for the point of order continuity, the norm is lower semicontinuous with respect to the convergence a.e.

Section 4 concerns the Lorentz spaces $\Gamma_{p,w}$ and $\Lambda_{p,w}$. We give the full characterization of H_g and H_l points. Several corollaries concerning respective global properties are also deduced.

In the last section, we show applications of H_g and H_l points to local best dominated approximation problems in Banach lattices. It is known that global monotonicity properties (strict and uniform monotonicity) play an analogous role in the best dominated approximation problems in Banach lattices as the respective rotundity properties (strict and uniform rotundity) do in the best approximation problems in Banach spaces (see [30]). The points of lower (upper) monotonicity of a Banach lattice E play an analogous role like the extreme points in a Banach space X. Similarly, the role of points of upper (lower) local uniform monotonicity in Banach lattices is analogous to that of points of local uniform rotundity in Banach spaces. The role of lower (upper) monotonicity points and points of order continuity in local best dominated approximation problems in Banach lattices has been investigated in [7]. We will show that although the order continuity and property H_g are not comparable each to other, H_g point has a similar impact in local best dominated approximation problems in Banach lattices as a point of order continuity. Recall that global properties of H_g and H_l points have been investigated among others in [10,22,23]. The uniform versions of these properties have been studied in [36].

2. Preliminaries

Let $\mathbb{R}$ and $\mathbb{N}$ be the sets of real and positive integers, respectively. As usual $S(X)$ (resp. $B(X)$) stands for the unit sphere (resp. the closed unit ball) of a Banach space $(X, \|\cdot\|_X)$.

Denote by L^0 the set of all (equivalence classes of) extended real valued Lebesgue measurable functions on $[0,\alpha)$, where $\alpha = 1$ or $\alpha = \infty$. Let m be the Lebesgue measure on $[0,\alpha)$.

A Banach lattice $(E, \|\cdot\|_E)$ is called a *Banach function space* (or a *Köthe space*) if it is a sublattice of L^0 satisfying the following conditions:

(1) if $x \in L^0$, $y \in E$ and $|x| \le |y|$ a.e., then $x \in E$ and $\|x\|_E \le \|y\|_E$;
(2) there exists a strictly positive element $x \in E$.

By E_+ we denote the positive cone of E, that is, $E_+ = \{x \in E : x \ge 0\}$. We use the notation $A^c = [0,\alpha)\backslash A$ for any measurable set A.

A point $x \in E$ is said to have an *order continuous norm* if for any sequence (x_n) in E such that $0 \le x_n \le |x|$ and $x_n \to 0$ m-a.e. we have $\|x_n\|_E \to 0$. A Köthe space E is called *order continuous* ($E \in (OC)$) if every element of E has an order continuous norm (see [20,31]). As usual E_a stands for the subspace of order continuous elements of E.

We will assume in the whole paper (unless it is stated otherwise) that E has the *Fatou property* ($E \in (FP)$), that is, if $0 \le x_n \uparrow x \in L^0$ with $(x_n)_{n=1}^\infty$ in E and $\sup_{n\in\mathbb{N}}\|x_n\|_E < \infty$, then $x \in E$ and