J. Math. Anal. Appl. 426 (2015) 734–746

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Two results on composition operators on the Dirichlet space

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ARTICLE INFO

*Article history:*
Received 17 October 2014
Available online 2 February 2015
Submitted by T. Ransford

*Keywords:*
Approximation numbers
Carleson embedding
Composition operator
Cusp map
Dirichlet space

ABSTRACT

We show that the decay of approximation numbers of compact composition operators on the Dirichlet space $\mathcal{D}$ can be as slow as we wish. We also prove the optimality of a result of O. El-Fallah, K. Kellay, M. Shabankhah and H. Youssfi on boundedness on $\mathcal{D}$ of self-maps of the disk all of whose powers are norm-bounded in $\mathcal{D}$.



## 1. Introduction

Recall that if $\varphi$ is an analytic self-map of $\mathbb{D}$, a so-called *Schur function*, the composition operator $C_\varphi$ associated to $\varphi$ is formally defined by

$$C_\varphi(f) = f \circ \varphi.$$

The Littlewood subordination principle [3, p. 30] tells us that $C_\varphi$ maps the Hardy space $H^2$ to itself for every Schur function $\varphi$. Also recall that if $H$ is a Hilbert space and $T\colon H \to H$ a bounded linear operator, the $n$-th approximation number $a_n(T)$ of $T$ is defined as

$$a_n(T) = \inf\big\{\|T - R\|;\ \text{rank } R < n\big\}, \quad n = 1, 2, \ldots. \tag{1.1}$$

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[1] Supported by a Spanish research project MTM 2012-05622.

http://dx.doi.org/10.1016/j.jmaa.2015.01.062

In [9], working on that Hardy space $H^2$ (and also on some weighted Bergman spaces), we have undertaken the study of approximation numbers $a_n(C_\varphi)$ of composition operators $C_\varphi$, and proved among other facts the following:

**Theorem 1.1.** *Let $(\varepsilon_n)_{n\geq 1}$ be a non-increasing sequence of positive numbers tending to 0. Then, there exists a compact composition operator $C_\varphi$ on $H^2$ such that*

$$\liminf_{n\to\infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.$$

*As a consequence, there are composition operators on $H^2$ which are compact but in no Schatten class.*

The last item had been previously proved by Carroll and Cowen [2], the above statement with approximation numbers being more precise.

For the Dirichlet space, the situation is more delicate because not every analytic self-map of $\mathbb{D}$ generates a bounded composition operator on $\mathcal{D}$. When this is the case, we will say that $\varphi$ is a *symbol* (understanding "of $\mathcal{D}$"). Note that every symbol is necessarily in $\mathcal{D}$.

In [8], we have performed a similar study on that Dirichlet space $\mathcal{D}$, and established several results on approximation numbers in that new setting, in particular the existence of symbols $\varphi$ for which $C_\varphi$ is compact without being in any Schatten class $S_p$. But we have not been able in [8] to prove a full analogue of Theorem 1.1. Using a new approach, essentially based on Carleson embeddings and the Schur test, we are now able to prove that analogue.

**Theorem 1.2.** *For every sequence $(\varepsilon_n)_{n\geq 1}$ of positive numbers tending to 0, there exists a compact composition operator $C_\varphi$ on the Dirichlet space $\mathcal{D}$ such that*

$$\liminf_{n\to\infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.$$

Turning now to the question of necessary or sufficient conditions for a Schur function $\varphi$ to be a symbol, we can observe that, since $(z^n/\sqrt{n}\,)_{n\geq 1}$ is an orthonormal sequence in $\mathcal{D}$ and since formally $C_\varphi(z^n) = \varphi^n$, a necessary condition is as follows:

$$\varphi \text{ is a symbol} \quad \Longrightarrow \quad \|\varphi^n\|_{\mathcal{D}} = O(\sqrt{n}\,). \tag{1.2}$$

It is worth noting that, for any Schur function, one has:

$$\varphi \in \mathcal{D} \quad \Longrightarrow \quad \|\varphi^n\|_{\mathcal{D}} = O(n)$$

(of course, this is an equivalence). Indeed, anticipating on the next section, we have for any integer $n \geq 1$:

$$\begin{aligned}\|\varphi^n\|_{\mathcal{D}}^2 &= |\varphi(0)|^{2n} + \int_{\mathbb{D}} n^2 |\varphi(z)|^{2(n-1)} |\varphi'(z)|^2 \, dA(z) \\ &\leq |\varphi(0)|^2 + \int_{\mathbb{D}} n^2 |\varphi'(z)|^2 \, dA(z) \leq n^2 \|\varphi\|_{\mathcal{D}}^2,\end{aligned}$$

giving the result.

Now, the following sufficient condition was given in [4]:

$$\|\varphi^n\|_{\mathcal{D}} = O(1) \quad \Longrightarrow \quad \varphi \text{ is a symbol.} \tag{1.3}$$