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General monotone functions and their Fourier coefficients

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ABSTRACT

We develop results on general monotone functions, and use these to extend classical results in harmonic analysis of Hardy and Littlewood. This also generalizes work of Askey and Boas, Sagher, and Liflyand and Tikhonov.

1. Notation

Let μ denote the Lebesgue measure. Let $E \subset \mathbf{C}$ be measurable. For $f(x)$, $\omega(x)$ measurable on E, with $\omega(x) > 0$ for $x \in E$, for $0 < q < \infty$, we denote

$$\|f\|_{L^q_\omega(E)} = \left(\int\limits_E \big(\omega(x)\big|f(x)\big|\big)^q \, dx \right)^{\frac{1}{q}}$$

and for $q = \infty$, we denote

$$\|f\|_{L^\infty_\omega(E)} = \operatorname*{ess\,sup}_{x \in E} \omega(x)\big|f(x)\big|.$$

For $1 \le q \le \infty$, these define norms. For $0 < q < 1$, these define seminorms; by abuse of language, we refer to them as norms. The weighted L^q-spaces are defined for $0 < q \le \infty$:

$$L^q_\omega(E) = \big\{ f : \|f\|_{L^q_\omega(E)} < \infty \big\}.$$

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For $E \subset (0, \infty)$, we are particularly interested in

$$\omega(x) = x^{\frac{1}{p} - \frac{1}{q}}$$

and with this weight, we denote for $0 < p < \infty$, $0 < q < \infty$:

$$\|f\|_{L^q_{\omega(p,q)}(E)} = \left(\int_E \left(x^{\frac{1}{p} - \frac{1}{q}} |f(x)| \right)^q dx \right)^{\frac{1}{q}} = \left(\int_E \left(x^{\frac{1}{p}} |f(x)| \right)^q \frac{dx}{x} \right)^{\frac{1}{q}}$$

while for $0 < p < \infty$, $q = \infty$, we denote

$$\|f\|_{L^\infty_{\omega(p,\infty)}(E)} = \operatorname*{ess\,sup}_{x \in E} x^{\frac{1}{p}} |f(x)|.$$

For $p = q = \infty$, we let

$$\|f\|_{L^\infty_{\omega(\infty,\infty)}(E)}$$

denote the usual L^∞-norm of f on E. The $L^q_{\omega(p,q)}$-spaces are defined for $0 < p < \infty$, $0 < q \leq \infty$, or $p = q = \infty$:

$$L^q_{\omega(p,q)}(E) = \left\{ f : \|f\|_{L^q_{\omega(p,q)}(E)} < \infty \right\}.$$

For a function f measurable and finite a.e. on $E \subset \mathbf{C}$, let f^* denote the nonincreasing rearrangement of $|f|$; that is, f^* is nonincreasing on $(0, \mu(E))$, and for all $\alpha > 0$,

$$\mu\{f^* > \alpha\} = \mu\{|f| > \alpha\}.$$

Denote for $0 < p < \infty$, $0 < q < \infty$:

$$\|f\|_{L(p,q)(E)} = \left(\int_0^{\mu(E)} \left(x^{\frac{1}{p} - \frac{1}{q}} f^*(x) \right)^q dx \right)^{\frac{1}{q}} = \left(\int_0^{\mu(E)} \left(x^{\frac{1}{p}} f^*(x) \right)^q \frac{dx}{x} \right)^{\frac{1}{q}}$$

and for $0 < p < \infty$, $q = \infty$:

$$\|f\|_{L(p,\infty)(E)} = \operatorname*{ess\,sup}_{0 < x < \mu(E)} x^{\frac{1}{p}} f^*(x).$$

In fact, it can be shown that

$$\|f\|_{L(p,\infty)(E)} = \sup_{0 < x < \mu(E)} x^{\frac{1}{p}} f^*(x).$$

For $p = q = \infty$, we let

$$\|f\|_{L(\infty,\infty)(E)}$$

denote the usual L^∞-norm of f on E. For $0 < p < \infty$, $0 < q \leq \infty$, these define seminorms; by abuse of language, we refer to them as norms. The Lorentz spaces $L(p, q)$ are defined for $0 < p < \infty$, $0 < q \leq \infty$, or $p = q = \infty$:

$$L(p,q)(E) = \left\{ f : \|f\|_{L(p,q)(E)} < \infty \right\}.$$