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On magnitude orderings between smallest order statistics from heterogeneous beta distributions

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ABSTRACT

In this work, we pay attention to smallest order statistics arising from heterogeneous random variables satisfying the proportional reversed hazard rate (PRHR) model. In particular, we show various results on this model related to comparisons based on magnitude stochastic orders. Since beta distribution is the simplest element of PRHR models, we first face the problem of some stochastic comparisons under some majorization conditions on the beta parameters. We then extend these results to the general case of PRHR models.

1. Introduction

Gupta et al. [5] introduced the proportional reversed hazard rate (PRHR) model as a dual of the well known proportional hazard rate (PHR) model. Given a random variable X with cumulative distribution function (cdf) F, the PRHR model is defined as

$$G_\alpha(x) = [F(x)]^\alpha ,$$

where $\alpha > 0$ is the constant of proportionality. In these models, the reversed hazard rate function associated with the cdf G_α is proportional to the reversed hazard rate function associated with the cdf F with constant of proportionality α, that is, $r_G = \alpha r_F$. When $F(x) = x$ for $0 < x < 1$, it is clear that G_α follows a beta distribution with shape parameters $\alpha > 0$ and $\gamma = 1$.

PRHR models are also known as exponentiated type distributions. For instance, the exponentiated exponential distribution, where F is the cdf of an exponential distribution, was defined by Gupta et al. [5]. Mudholkar and Srivastava [12] proposed the exponentiated Weibull distribution to analyze bathtub failure data, in this case, F is the cdf of a Weibull distribution. Nadarajah and Kotz [13] introduced three more exponentiated type distributions: the exponentiated gamma, the exponentiated Gumbel and the exponentiated Fréchet distributions, where F is the cdf of a gamma, Gumbel and Fréchet distributions, respectively.

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Methods of inference of the PRHR model are studied in [6] and [4], among others. Relating to applications, the PRHR model is appropriate to many real practical situations in different scientific areas. For instance, in survival data analysis, the PRHR model plays the same role in the analysis of left-censored data as the PHR model plays in the analysis of right-censored data (see, e.g., [14]).

Stochastic comparisons as well as aging properties in this model were investigated, in the univariate case, by Di Crescenzo [2] and, in the multivariate case, by Li and Da [10]. Stochastic order relations between extreme order statistics arising from two vectors of independent random variables belonging to the PRHR model have been studied in [3], when the variables in one set have proportionality parameters $(\alpha_1, \ldots, \alpha_n)$ and in the other set are independent and identically distributed random variables with common parameter $\bar{\alpha} = \sum_{i=1}^{n} \alpha_i/n$. Recently, Balakrishnan et al. [1] proved the usual multivariate stochastic ordering for two sets of independent but not identically distributed random variables satisfying the PRHR model in a sample of size two.

The purpose of this work is to compare the smallest order statistics of two sets of heterogeneous random variables arising from PRHR models in the sense of hazard rate, reversed hazard rate and likelihood ratio orderings when the vectors of proportionality parameters are ordered in some majorization order. To do this, since beta distribution is the simplest element of PRHR models, we first present some new results concerning stochastic properties between the smallest order statistics arising from heterogeneous beta distributions. These results are also of independent interest.

The rest of the article is organized as follows. In Section 2, we review various types of stochastic and majorization orders and give some auxiliary results. In Sections 3 and 4, we investigate stochastic orderings between the smallest order statistics arising from heterogeneous beta distributions and from PRHR models, respectively. Finally, in Section 5, we discuss possible extensions of this work and some open problems.

2. Definitions and preliminary results

In this section, we first recall some notions of stochastic and majorization orderings. Throughout the article the term *increasing* (*decreasing*) is used for monotone *non-decreasing* (*non-increasing*).

Let X and Y be univariate random variables with cumulative distribution functions (cdf's) F and G, survival functions $\bar{F}\,(= 1 - F)$ and $\bar{G}\,(= 1 - G)$, pdf's f and g, hazard rate functions $h_F\,(= f/\bar{F})$ and $h_G\,(= g/\bar{G})$, and reversed hazard rate functions $r_F\,(= f/F)$ and $r_G\,(= g/G)$, respectively. The following definitions introduce stochastic orders, which are considered in this article, to compare the magnitudes of two random variables. For more details on stochastic comparisons, see [15].

Definition 2.1. We say that X is smaller than Y in the:

a) usual stochastic order, denoted by $X \le_{\mathrm{st}} Y$, if $\bar{F}(x) \le \bar{G}(x)$ for all x,
b) hazard rate order, denoted by $X \le_{\mathrm{hr}} Y$, if $\bar{G}(x)/\bar{F}(x)$ increases in x. If X and Y are absolutely continuous, then $X \le_{\mathrm{hr}} Y$ is equivalent to $h_F(x) \ge h_G(x)$ for all x,
c) reversed hazard rate order, denoted by $X \le_{\mathrm{rh}} Y$, if $G(x)/F(x)$ increases in x. If X and Y are absolutely continuous, then $X \le_{\mathrm{rh}} Y$ is equivalent to $r_F(x) \le r_G(x)$ for all x,
d) likelihood ratio order, denoted by $X \le_{\mathrm{lr}} Y$, if $g(x)/f(x)$ is increasing in x for which the ratio is well defined.

We shall also be using the concept of majorization in our discussion. Let $\{x_{(1)}, x_{(2)}, \ldots, x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$.

Definition 2.2. The vector $\boldsymbol{x}$ is said to be majorized by the vector $\boldsymbol{y}$, denoted by $\boldsymbol{x} \overset{\mathrm{m}}{\le} \boldsymbol{y}$, if

$$\sum_{i=1}^{j} x_{(i)} \ge \sum_{i=1}^{j} y_{(i)}, \quad \text{for } j = 1, \ldots, n-1 \quad \text{and} \quad \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$