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Multiple solutions for a class of Schrödinger–Poisson system with indefinite nonlinearity

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ABSTRACT

In this article, we investigate the existence and multiplicity of solutions of Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + l(x)\phi u = k(x)|u|^2 u + \lambda h(x) u, & x \in \mathbb{R}^3 \\ -\Delta \phi = l(x) u^2, & x \in \mathbb{R}^3 \end{cases}$$

where the potential $k(x)$ allows sign changing. We obtain the existence and multiplicity of solutions for the system, which can be regarded as complementary work of Huang et al. (2013) [12].

1. Introduction

The system

$$\begin{cases} -\Delta u + u + l(x)\phi u = g(x,u) & x \in \mathbb{R}^3 \\ -\Delta \phi = l(x) u^2 & x \in \mathbb{R}^3 \end{cases} \tag{1.1}$$

arises in quantum mechanics models and semiconductor theory. Many researchers have considered the system mainly in the autonomous case: see e.g. [3,13]. By Pohozaev equality or Nehari manifold method, they proved existence and nonexistence of solutions of the system (see [10]). By combining Nehari manifold and Pohozaev equality, Ruiz [13] obtained a certain manifold and considered the minimizing energy functional J on it. He proved some existence and nonexistence results. However, the method is hardly applied to the non-autonomous case. In the present paper, we consider the non-autonomous case that $g(x,u)$ is a combination of a 4-linear term and a linear term. More precisely, we study the following system with the form

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$$\begin{cases} -\Delta u + u + l(x)\phi u = k(x)|u|^{p-2}u + \lambda h(x)u & x \in \mathbb{R}^3 \\ -\Delta \phi = l(x)u^2 & x \in \mathbb{R}^3 \end{cases} \tag{SP}$$

where $p = 4$ and $\lambda > 0$. As far as we know, no one considered this case before. For non-autonomous case, many mathematicians mainly considered the case $4 < p < 6$ or $p = 2$ see [8,5,14]. Specifically, Cerami and Vaira [5] studied the case $g(x, u) = k(x)|u|^{p-2}u$ where $4 < p < 6$. Sun et al. [14] considered the case $p = 2$, i.e. they considered the case of asymptotically linear at infinity. More recently, Huang et al. [12] considered the case $g(x, u) = k(x)|u|^{p-2}u + \lambda h(x)u$ where $4 < p < 6$. They proved existence and multiplicity results by some ideas developed in [9]. To do this they used the mountain pass theorem of [2]. Motivated by [12,14,5], we are interested in the case that $p = 4$ and $k(x)$ is sign changing on $\mathbb{R}^3$. It is well known that the system (SP) can be easily transformed to a nonlinear Schrödinger equation with a non-local term $\int_{\mathbb{R}^3} l(x)\phi_u(x)u^2$ (see later). Since the non-local term $\int_{\mathbb{R}^3} l(x)\phi_u(x)u^2$ is 4-order and $k(x)$ is sign changing on $\mathbb{R}^3$, the geometrical structure of mountain pass may fail if $p = 4$. Moreover, under our hypotheses, whether the (PS) property holds or not remains incognito. Therefore, we cannot simply apply the dual method from [2]. It means that the methods used in above papers may not work anymore. In order to state our main results, we assume the following hypotheses (H):

(H_h) $h \in L^{\frac{3}{2}}(\mathbb{R}^3)$, $h(x) \geq 0$ for any $x \in \mathbb{R}^3$ and $h \not\equiv 0$;
(H_{k_1}) $k(x) \in C(\mathbb{R}^3)$ and $k(x)$ changes sign in $\mathbb{R}^3$;
(H_{k_2}) $\lim_{|x|\to\infty} k(x) = k_\infty < 0$;
(H_l) $l(x) \geq 0$, $l(x) \in L^\infty(\mathbb{R}^3)$ or $l(x) \in L^2(\mathbb{R}^3)$.

Under hypothesis (H_h), there exists a sequence of eigenvalues λ_n of

$$-\Delta u + u = \lambda h(x)u \quad \text{in } H^1(\mathbb{R}^3)$$

with $0 < \lambda_1 < \lambda_2 \leq \cdots$ and each eigenvalue being of finite multiplicity. The associated normalized eigenfunctions are denoted by $e_1, e_2, \cdots$ with $\|e_i\| = 1$. Moreover, $e_1 > 0$ in $\mathbb{R}^3$.

We are now ready to state our results:

Theorem 1.1. *Assume hypotheses (H) hold. Then for $0 < \lambda < \lambda_1$, problem (SP) has at least one solution in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.*

Theorem 1.2. *Assume hypotheses (H) hold and suppose $\int_{\mathbb{R}^3} k(x){e_1}^4 - l(x)\phi_{e_1}{e_1}^2 < 0$. Then there exists $\delta > 0$ such that problem (SP) has at least two solutions whenever $\lambda_1 < \lambda < \lambda_1 + \delta$.*

Remark 1. In [12], the authors proved similar results for $4 < p < 6$, but they did not need the condition $\int_{\mathbb{R}^3} k(x){e_1}^4 - l(x)\phi_{e_1}{e_1}^2 < 0$. However, they need additional conditions:

(H_{l_1}) $l(x) \in L^2(\mathbb{R}^3), l(x) \geq 0$ for any $x \in \mathbb{R}^3$ and $l \not\equiv 0$.
(H_{l_2}) $l(x) = 0$ a.e. in $\Omega^0 = \{x \in \mathbb{R}^3 : k(x) = 0\}$ and Ω_0 coincides with the closure of its interior.

Hypothesis (H_{l_1}) insures that the functional $u \mapsto \int_{\mathbb{R}^3} l(x)\phi_u(x)u^2 dx$ is weakly continuous. It plays an importance role in the proof of PS property.

Remark 2. To the best of our knowledge, a similar condition like $\int_{\mathbb{R}^3} k(x){e_1}^q dx < 0$ is needed for the semilinear elliptic equations with indefinite nonlinearity, (see [1,5,4] and so on). Obviously, if $\int_{\mathbb{R}^3} k(x){e_1}^4 dx < 0$, our hypotheses hold.