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Necessary condition for compactness of a difference of composition operators on the Dirichlet space

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ABSTRACT

Let φ be a self-map of the unit disk and let C_φ denote the composition operator acting on the standard Dirichlet space $\mathcal{D}$. A necessary condition for compactness of a difference of two bounded composition operators acting on $\mathcal{D}$ is given. As an application, a characterization of disk automorphisms φ and ψ, for which the commutator $[C_\psi^*, C_\varphi]$ is compact, is given.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ denote the open unit disk in the complex plane $\mathbb{C}$ and let $\mathbb{T} = \{z : |z| = 1\}$ denote the unit circle in $\mathbb{C}$. The Dirichlet space $\mathcal{D}$ is the space of all analytic functions f in $\mathbb{D}$, such that

$$\|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty,$$

where $dA(z) = \pi^{-1} dxdy$ is the normalized two dimensional Lebesgue measure on $\mathbb{D}$. The Dirichlet space is a Hilbert space with inner product

$$\langle f, g\rangle_{\mathcal{D}} := f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} dA(z).$$

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The Dirichlet space has the reproducing kernel property and the kernel function is defined as

$$K_w(z) := 1 + \log \frac{1}{1 - \overline{w}z}, \tag{1.1}$$

where the branch of the logarithm is chosen such that

$$\log \frac{1}{1 - \overline{w}z} = \sum_{n=1}^{\infty} \frac{(\overline{w}z)^n}{n}.$$

By a self-map of $\mathbb{D}$ we mean an analytic function φ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. We will also assume that a self-map φ is not a constant function. For a self-map of the unit disk φ, the composition operator C_φ on the Dirichlet space $\mathcal{D}$ is defined by $C_\varphi f := f \circ \varphi$. The composition operator C_φ on Dirichlet space is not necessarily bounded for an arbitrary self-map of the unit disk. However, C_φ is bounded on $\mathcal{D}$ if, for example, φ is a finitely valent function (see, e.g., [9,13]). More is known about the composition operator C_φ when the symbol φ is a linear-fractional self-map of the unit disk of the form

$$\varphi(z) := \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$. In that case C_φ is compact on $\mathcal{D}$ if and only if $\|\varphi\|_\infty < 1$ (see, e.g., [3,11,13]).

For an arbitrary self-map of the unit disk φ, if the operator C_φ is bounded, then the adjoint operator C_φ^* satisfies

$$C_\varphi^* f(w) = \langle f, K_w \circ \varphi \rangle_{\mathcal{D}},$$

which yields a useful equality

$$C_\varphi^* K_w = K_{\varphi(w)}. \tag{1.2}$$

For φ a linear-fractional self-map of $\mathbb{D}$, Gallardo-Gutiérrez and Montes-Rodríguez in [4] (see also [8]) proved that the adjoint of the composition operator is given by formula

$$C_\varphi^* f = f(0)K_{\varphi(0)} - (C_{\varphi^*} f)(0) + C_{\varphi^*} f, \tag{1.3}$$

where

$$\varphi^*(z) := \frac{1}{\overline{\varphi^{-1}\left(\frac{1}{\bar{z}}\right)}}, \quad z \in \mathbb{D},$$

is the Krein adjoint of φ. It is worth to note that φ^* is a linear-fractional self-map of the unit disk, in fact

$$\varphi^*(z) = \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}}.$$

It is easy to check that w is a fixed point of φ if and only if $1/\overline{w}$ is a fixed point of φ^*. In particular, if φ has a fixed point on $\mathbb{T}$, then it is a fixed point of both φ and φ^*.

Let φ be a disk automorphism, which is of the form

$$\varphi(z) = e^{i\theta} \frac{a - z}{1 - \bar{a}z}, \quad z \in \mathbb{D}, \tag{1.4}$$

where $a \in \mathbb{D}$ and $\theta \in (-\pi, \pi]$. We will say that