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Periodic van der Pol equation with state dependent impulses

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A B S T R A C T

Conditions are given for the existence of a solution of given period to the impulsively driven van der Pol equation. Results are obtained for impulses of varying degrees of state dependence.

1. Introduction

For given $\mu > 0$, $T > 0$ and an almost everywhere T-periodic real function $f : \mathbb{R} \to \mathbb{R}$ Lebesgue integrable on $[0, T]$, we consider the impulsively driven Liénard equation

$$x'' - \mu(x - \frac{x^3}{3})' + x = f + \sum_{i=1}^{m} a_i[x]\delta_{t_i[x]} \tag{1}$$

where the impulses $\delta_{t_i[x]}$ have state dependent amplitude $a_i[x] \in \mathbb{R}$ at state dependent instants $t_i[x] + kT$ ($k \in \mathbb{Z}$, $t_i \in [0, T)$, $i \in \{1, \dots, m\}$). This equation, in the absence of impulsive forcing, was introduced by van der Pol [20–22] to model a vacuum tube triode circuit. Cartwright and Littlewood studied the non impulsive forced equation in [5,6,15,16], as did also more recently Guckenheimer, Hoffman and Weckesser in [10], Kalas and Kadeřábek in [11] and Lin in [14] (for example). In this paper we approach the problem of the existence of T-periodic solutions of (1). Such results already exist in the literature for some first and second order nonlinear differential equations. (See [1–3,7,8,17].) In the case of (1), the expression $(x - x^3/3)'$ fails to satisfy a simple Lipschitz condition, which adds to the difficulty in obtaining similar results.

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We write $L^1(T)$ to denote the Banach space of almost everywhere T-periodic functions $x : \mathbb{R} \to \mathbb{R}$ with norm

$$\|x\|_1 = \frac{1}{T}\int_0^T |x(t)|\,dt$$

and $NBV(T)$ for the family of all $x \in L^1(T)$ which are of bounded variation over $[0,T]$ and normalized in the sense that

$$x(t) = \frac{1}{2}[x(t-) + x(t+)]$$

for all $t \in \mathbb{R}$. We shall obtain conditions that guarantee the existence of a solution $x \in NBV(T)$ of (1) such that $x' \in NBV(T)$ and so such that $x'' \in L^1(T)$ [19, p. 157]. First we shall consider the state independent case (where a_i and t_i are constants for all i). This will be followed by the mean-state dependent case. The general state dependent case will be considered last. We found that this order of presentation made the paper easier to follow, without affecting its length.

2. Preliminaries

Any T-periodic generalized function $x(t)$ can be identified with a Fourier series by the formula

$$x(t) = \sum_{n\in\mathbb{Z}} \hat{x}(n)e^{in\omega t} \quad (\omega = 2\pi/T)$$

for $\hat{x}(n) \in \mathbb{C}$. Its generalized derivative, denoted $x'(t)$, is also identified with a Fourier series by way of the formula

$$x'(t) = \sum_{n\in\mathbb{Z}} in\omega\hat{x}(n)e^{in\omega t}.$$

The mean $\bar{x}$ of x is given by

$$\bar{x} = \hat{x}(0)$$

and we define $\tilde{x}$ by

$$\tilde{x} = x - \bar{x}.$$

In particular (see, for example, [23, p. 333])

$$\delta_{t'}(t) = \sum_{n\in\mathbb{Z}} e^{in\omega(t-t')},$$

$$\overline{\delta_{t'}} = 1$$

and

$$\widetilde{\delta_{t'}}(t) = \sum_{n\in\mathbb{Z}\setminus\{0\}} e^{in\omega(t-t')}$$