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A note on boundary layer of a nonlinear evolution system with damping and diffusions

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A B S T R A C T

In this paper, we consider an initial-boundary value problem for a nonlinear evolution system with damping and diffusions on the strip. Our main purpose is to investigate zero-diffusion limits, which formally yield parabolic–hyperbolic coupled equations resulting from parabolic–parabolic coupled equations. It is shown that the boundary layer thickness is of the order $O(\beta^\gamma)$ with $0 < \gamma < \frac{3}{4}$ as the diffusion parameter β goes to zero.

Contents

1. Introduction

The boundary-layer effect is a well known phenomenon in hydrodynamics. The theory of boundary layers has been one of the fundamental and important issues in fluid dynamics since it was proposed by Prandtl in 1904, cf. [27]. It occurs when solutions of the Navier–Stokes equations do not tend asymptotically

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to solutions of the Euler equations near boundaries as the viscosity vanishes. For example, there are a number of papers dedicated to the questions of boundary layers for both incompressible and compressible Navier–Stokes equations, cf. [10,11,16,20,25,31,33–35]. The question of boundary layer for other equation was also addressed in a number of works such as [17].

The same effect also arises in the theory of hyperbolic systems when parabolic equations with small viscosity are applied as perturbations; see, for instance, [12,13,29] and the references cited therein. The rigorous mathematical description of the asymptotic behavior of viscous solutions near boundaries for the small viscosity is a challenging problem. In general, such singular perturbation problems require introducing "boundary layers", but there are noteworthy problems when there is no need of boundary layers, cf. [2,3].

In this paper, we consider the following initial-boundary value problem for the nonlinear evolution system with damping and diffusions on the strip $[0,1]\times[0,\infty)$

$$\begin{cases} \psi_t^\beta = -(\sigma-\alpha)\psi^\beta - \sigma\theta_x^\beta + \alpha\psi_{xx}^\beta, \\ \theta_t^\beta = -(1-\beta)\theta^\beta + \nu\psi_x^\beta + 2\psi^\beta\theta_x^\beta + \beta\theta_{xx}^\beta, \quad 0<x<1,\ t>0, \end{cases} \tag{1.1}$$

with initial data

$$\left(\psi^\beta,\theta^\beta\right)(x,0) = (\psi_0,\theta_0)(x), \quad 0\le x\le 1, \tag{1.2}$$

and the boundary conditions

$$\left(\psi^\beta,\theta_x^\beta\right)(0,t) = \left(\psi^\beta,\theta_x^\beta\right)(1,t) = (0,0), \quad t\ge 0, \tag{1.3}$$

which implies

$$\left(\psi_t^\beta,\theta_{xt}^\beta\right)(0,t) = \left(\psi_t^\beta,\theta_{xt}^\beta\right)(1,t) = (0,0), \quad t\ge 0, \tag{1.4}$$

where we assume the following compatible condition is valid: $(\psi_0,\theta_{0x})(0) = (\psi_0,\theta_{0x})(1) = (0,0)$. Here α,β,σ and ν are all positive constants satisfying the relation $\alpha<\sigma$ and $0<\beta<1$. The system (1.1) was originally proposed by Hsieh in [14] to observe the nonlinear interaction between ellipticity and dissipation. We also refer to [14,18,19] for the physical background of (1.1).

The system (1.1) has been extensively studied by several authors in different contexts. However most of them need to assume that all parameters σ,α,β and ν are fixed constants, and we refer to [4,7,8,15,23,30,32,36] and references therein.

An interesting problem is the zero-diffusion limits, which formally yield parabolic–hyperbolic coupled equations resulting from parabolic–parabolic coupled equations. Moreover the boundary layer will occur due to the boundary effect. Such "singularly perturbed" problems are even found in kinetic equations like Boltzmann, Boltzmann–Poisson and Vlasov–Poisson–Fokker–Planck equations, cf. [1,5,9,22,26]. To our knowledge, very fewer results on the system (1.1) have been obtained in this direction, cf. [6,28].

Firstly, Chen and Zhu in [6] considered the Cauchy problem

$$\begin{cases} \psi_t = -(1-\alpha)\psi - \theta_x + \alpha\psi_{xx}, \\ \theta_t = -(1-\alpha)\theta - \mu\psi_x + 2\psi\theta_x + \alpha\theta_{xx}, \end{cases} \tag{1.5}$$

with initial data

$$\left(\psi^\alpha,\theta^\alpha\right)(x,0) = (\psi_0,\theta_0)(x) \to (\psi_\pm,\theta_\pm) \quad \text{as } x\to\pm\infty, \tag{1.6}$$

where α and μ are positive constants satisfying $0<\alpha<1$. They proved that the solution sequences $\{(\psi^\alpha,\theta^\alpha)\}$ of (1.5), (1.6) converge to the solution to the corresponding limit system with $\alpha=0$ as $\alpha\to 0^+$. In their argument, $\mu>0$ plays a key role.