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# One modification of the martingale transform and its applications to paraproducts and stochastic integrals

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A B S T R A C T

In this paper we introduce a variant of Burkholder's martingale transform associated with two martingales with respect to different filtrations. Even though the classical martingale techniques cannot be applied, we show that the discussed transformation still satisfies some expected $\mathrm{L}^p$ estimates. Then we apply the obtained inequalities to general-dilation twisted paraproducts, particular instances of which have already appeared in the literature. As another application we construct stochastic integrals $\int_0^t H_s d(X_s Y_s)$ associated with certain continuous-time martingales $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$. The process $(X_t Y_t)_{t\geq 0}$ is shown to be a "good integrator", although it is not necessarily a semimartingale, or even adapted to any convenient filtration.


## 1. Introduction and statement of the results

### 1.1. Discrete-time estimates

If $(U_k)_{k=0}^\infty$ and $(V_k)_{k=0}^\infty$ are two completely arbitrary discrete-time stochastic processes, let us agree to write $(U\cdot V)_{n=0}^\infty$ for a new process defined by

$$(U\cdot V)_n := \sum_{k=1}^{n} U_{k-1}(V_k - V_{k-1}) \tag{1.1}$$

and adopt the convention $(U\cdot V)_0 = 0$. In the particular case when $(V_k)_{k=0}^\infty$ is a martingale and $(U_k)_{k=0}^\infty$ is bounded and adapted with respect to the same filtration, the above process is precisely Burkholder's

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*martingale transform* [7]. It plays an important role in finding sharp estimates for singular integral operators [2], the theory of UMD spaces [10], and inequalities for stochastic integrals [8]. See [9] and [1] for more details and references to the extensive literature. However, here we consider a different setting, which is motivated by a probabilistic technique in the proof of boundedness of a certain two-dimensional paraproduct-type operator [19].

Let us begin by describing a special case of two filtrations $(\mathcal{F}_k)_{k=0}^{\infty}$ and $(\mathcal{G}_k)_{k=0}^{\infty}$ that will be used throughout this work. Suppose that the underlying probability space is the product $(\Omega_1 \times \Omega_2, \mathcal{A} \otimes \mathcal{B}, \mathbb{P}_1 \times \mathbb{P}_2)$ of two probability spaces $(\Omega_1, \mathcal{A}, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{B}, \mathbb{P}_2)$. Whenever we write $\mathbb{E}$ alone, it will be understood that the expectation is taken with respect to the product measure $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$. Similarly we do with the Lebesgue spaces and their norms. Suppose that we are also given two filtrations $(\mathcal{A}_k)_{k=0}^{\infty}$ and $(\mathcal{B}_k)_{k=0}^{\infty}$ of $\mathcal{A}$ and $\mathcal{B}$, respectively, and denote

$$\mathcal{F}_k := \mathcal{A}_k \otimes \mathcal{B}, \quad \mathcal{G}_k := \mathcal{A} \otimes \mathcal{B}_k \tag{1.2}$$

for each nonnegative integer $k$. We can think of $(\mathcal{F}_k)_{k=0}^{\infty}$ and $(\mathcal{G}_k)_{k=0}^{\infty}$ as being a "horizontal" and a "vertical" filtration of $\mathcal{A} \otimes \mathcal{B}$, respectively. We remark that the two filtrations in (1.2) are not necessarily independent — in fact, they rarely are. Proposition 7 in the closing section will help us develop the intuition by showing that sigma algebras $\mathcal{F}_k$ and $\mathcal{G}_\ell$ are indeed independent conditionally on $\mathcal{F}_k \cap \mathcal{G}_\ell$.

Suppose that $(X_k)_{k=0}^{\infty}$ is a real-valued martingale with respect to the filtration $(\mathcal{F}_k)_{k=0}^{\infty}$ and that $(Y_k)_{k=0}^{\infty}$ is a real-valued martingale with respect to $(\mathcal{G}_k)_{k=0}^{\infty}$. Finally, let $(K_k)_{k=0}^{\infty}$ be an adapted process with respect to the filtration $(\mathcal{F}_k \cap \mathcal{G}_k)_{k=0}^{\infty}$. For processes $((KX \cdot Y)_n)_{n=0}^{\infty}$ and $((K \cdot XY)_n)_{n=0}^{\infty}$ our Definition (1.1) unfolds as

$$(KX \cdot Y)_n = \sum_{k=1}^{n} K_{k-1} X_{k-1} (Y_k - Y_{k-1}),$$

$$(K \cdot XY)_n = \sum_{k=1}^{n} K_{k-1} (X_k Y_k - X_{k-1} Y_{k-1}).$$

These processes are no longer adapted to any convenient filtration, so they cannot be treated in the same way as Burkholder's transform. We further discuss those difficulties in Section 6. Nevertheless, they still prove to be useful and they still satisfy some $\mathrm{L}^p$ estimates.

Let us adopt the notation $\|U\|_{\mathrm{L}^p} := (\mathbb{E}|U|^p)^{1/p}$ for any random variable $U$ and $1 \le p < \infty$, while $\|U\|_{\mathrm{L}^\infty}$ is simply defined as the essential supremum of $|U|$. For some of the desired estimates we will need that the intersection of filtrations satisfies the following "uniform growth" property: there exists a constant $A$ such that

$$\big\|\mathbb{E}(U|\mathcal{F}_{k+1} \cap \mathcal{G}_{k+1})\big\|_{\mathrm{L}^\infty} \le A \big\|\mathbb{E}(U|\mathcal{F}_k \cap \mathcal{G}_k)\big\|_{\mathrm{L}^\infty} \tag{1.3}$$

for any random variable $U \ge 0$ and any integer $k \ge 0$.

**Theorem 1.**

(a) *There exists an absolute constant $C$ such that for each nonnegative integer $n$ we have the inequalities:*

$$\|(KX \cdot Y)_n\|_{\mathrm{L}^{4/3}} \le C \|X_n\|_{\mathrm{L}^4} \|(K \cdot Y)_n\|_{\mathrm{L}^2}, \tag{1.4}$$

$$\|(K \cdot XY)_n\|_{\mathrm{L}^{4/3}} \le C \big( \|X_n\|_{\mathrm{L}^4} \|(K \cdot Y)_n\|_{\mathrm{L}^2} + \|Y_n\|_{\mathrm{L}^4} \|(K \cdot X)_n\|_{\mathrm{L}^2} \big). \tag{1.5}$$