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On an inverse boundary value problem of a nonlinear elliptic equation in three dimensions

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ABSTRACT

This work considers an inverse boundary value problem for a 3D nonlinear elliptic partial differential equation in a bounded domain. In general, the problem is severely ill-posed. The formal solution can be written as a hyperbolic cosine function in terms of the 2D elliptic operator via its eigenfunction expansion, and it is shown that the solution is stabilized or regularized if the large eigenvalues are cut off. In a theoretical framework, a truncation approach is developed to approximate the solution of the ill-posed problem in a regularization manner. Under some assumptions on regularity of the exact solution, we obtain several explicit error estimates including an error estimate of Hölder type. A local Lipschitz case of source term for this nonlinear problem is obtained. For numerical illustration, two examples on the elliptic sine-Gordon and elliptic Allen–Cahn equations are constructed to demonstrate the feasibility and efficiency of the proposed methods.

1. Introduction

In this paper, we consider the problem of reconstructing the temperature of a body from interior measurements. In fact, in many engineering contexts (see, e.g., [4]), we cannot attach a temperature sensor at the surface of a body (e.g., the skin of a missile). Hence, to get the temperature distribution on the surface, we have to use the temperature measured inside the body. Let L be a positive real number and $\Omega = (0, \pi) \times (0, \pi)$. We are interested in the following inverse boundary value problem: Find $u(x, y, 0)$ for $(x, y) \in \Omega$ where $u(x, y, z)$ satisfies the following nonlinear elliptic equation:

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$$\Delta u = F\big(x, y, z, u(x, y, z)\big), \quad (x, y, z) \in \Omega \times (0, +\infty), \tag{1.1}$$

subject to the conditions

$$\begin{cases} u(x, y, z) = 0, & (x, y, z) \in \partial\Omega \times (0, +\infty), \\ u(x, y, L) = \varphi(x, y), & (x, y) \in \Omega, \\ u_z(x, y, L) = 0, & (x, y) \in \Omega. \end{cases} \tag{1.2}$$

Here $\Delta u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2$, the function $\varphi \in L^2(\Omega)$ is known, and F is called the source function to be defined later. Having found $u(x, y, 0)$ a forward problem can be solved to find $u(x, y, z)$ for all $(x, y, z) \in \Omega \times (0, L)$.

It is widely recognized nowadays that Cauchy problems for the Poisson equation, and more generally for elliptic equations, have a central position in all inverse boundary value problems which are encountered in many practical applications such as electrocardiography [18], astrophysics [6] and plasma physics [3,16]. These problems are also closely related to inverse source problems arising from, e.g., electroencephalography and magnetoencephalography [19]. The continued interest in this kind of problems is evidenced by the number of publications on this topic. We refer to the monograph [18] for further reading on Cauchy problems for elliptic equations.

It is well-known that inverse boundary value problems are exponentially ill-posed in the sense of Hadamard. Existence of solutions and their stability with respect to given data do not hold even if the data are very smooth. In fact, the problems are extremely sensitive to measurement errors; hence, even in the case of existence, a solution does not depend continuously on the given data. This, of course, implies that a properly designed numerical treatment is required.

Inverse boundary problems for *linear* elliptic equations have been studied extensively, see, e.g., [1,3]. Indeed, in the case $F = 0$ in (1.1) with the following conditions

$$\begin{cases} u(x, y, z) = 0, & (x, y, z) \in \partial\Omega \times (0, +\infty), \\ u(x, y, L) = \varphi(x, y), & (x, y) \in \Omega, \\ \lim\limits_{z\to\infty} u_z(x, y, L) = 0, & (x, y) \in \Omega, \end{cases} \tag{1.3}$$

the problem is studied in [7,19,24]. In these studies, the algebraic invertibility of the inverse problem is established. However, regularization is not investigated. In [17], the authors apply the nonlocal boundary value method to solve an abstract Cauchy problem for the homogeneous elliptic equation. Eldén et al. develop useful numerical methods to solve the homogeneous problem; see for example [10–13]. Level set type methods are also proposed [23] for Cauchy problems for linear elliptic equations.

Although there are many works on Cauchy problems for linear elliptic equations, to the best of our knowledge the literature on the *nonlinear* case is very few. In the abstract framework of operators on Hilbert spaces, regularization techniques are developed by B. Kaltenbacher and her coauthors in [2,20–22]. The present paper serves to develop necessary theoretical bases for a regularization of problem (1.1)–(1.2).

Our approach can be summarized as follows. Let φ and φ^ϵ be the exact and measured data at $z = L$, respectively, which satisfy $\|\varphi - \varphi^\epsilon\|_{L^2(\Omega)} \le \epsilon$. Assume that problem (1.1)–(1.2) has a unique solution $u(x, y, z)$. By using the method of separation of variables, one can show that

$$u(x, y, z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\cosh\big((L-z)\sqrt{m^2+n^2}\big)\varphi_{mn} + \int_z^L \frac{\sinh((\tau - z)\sqrt{m^2+n^2})}{\sqrt{m^2+n^2}} F_{mn}(u)(\tau)d\tau\right]\phi_{mn}(x, y). \tag{1.4}$$