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Korn inequality on irregular domains

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ABSTRACT

In this paper, we study the weighted Korn inequality on some irregular domains, e.g., s-John domains and domains satisfying quasihyperbolic boundary conditions. Examples regarding sharpness of the Korn inequality on these domains are presented. Moreover, we show that Korn inequalities imply certain Poincaré inequality.

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1. Introduction

Let $p > 1$ and Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 2$. For each vector $\mathbf{v} = (v_1, \cdots, v_n) \in W^{1,p}(\Omega)^n$, let $D\mathbf{v}$ denote its gradient matrix, and $\epsilon(\mathbf{v})$ denote the symmetric part of $D\mathbf{v}$, i.e., $\epsilon(\mathbf{v}) = (\epsilon_{i,j}(\mathbf{v}))_{1\leq i,j\leq n}$ with

$$\epsilon_{i,j}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

Korn's (second) inequality states that, if Ω is sufficient regular (e.g., Lipschitz), then there exists $C > 0$ such that

$$\int_\Omega |D\mathbf{v}|^p\,dx \leq C\left\{\int_\Omega \left|\epsilon(\mathbf{v})\right|^p dx + \int_\Omega |\mathbf{v}|^p\,dx\right\}. \tag{K_p}$$

The Korn inequality (K_p) is a fundamental tool in the theory of linear elasticity equations; see [3,1,7,9,11,13,22,30] and the references therein. Notice that Korn's inequality (K_p) fails for $p = 1$ even on a cube; see the example from [6].

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On $\mathbb{R}^2$ and $p = 2$, several different inequalities (including the Friedrichs inequality) are actually equivalent to Korn's inequality (K_p) on simply connected Lipschitz domains; see [14,30] for example.

Friedrichs [11] proved the Korn inequality (K_p) for $p = 2$ on domains with a finite number of corners or edges on $\partial\Omega$, Nitsche [27] proved the Korn inequality (K_p) for $p = 2$ on Lipschitz domains, while Mjasnikov and Mosolov [26] and Ting [31] proved (K_p) for all $p \in (1, \infty)$; Kondratiev and Oleinik [22] studied the Korn inequality (K_2) on star-shaped domains. Recently, Acosta, Durán and Muschietti [3] proved the Korn inequality (K_p) holds for all $p \in (1, \infty)$ on John domains.

Weighted Korn inequality on irregular domains (in particular, Hölder domains) has received considerable interest recently; see [3,1,2,7,22] and references therein. Motivated by this, in this paper, we study weighted Korn inequality on some irregular domains including s-John domains ($s \geq 1$) and domains satisfying quasi-hyperbolic boundary conditions.

It is well known that one can deduce the (weighted) Korn inequality from the divergence equation, see [7,8] for instance. In particular, it was shown in [8, Proposition 3.2] that the validity of Poincaré inequality

$$\int_\Omega \left|u(x) - u_\Omega\right|^p dx \leq C \int_\Omega \left|\nabla u(x)\right|^p \operatorname{dist}(x, \partial\Omega)^b \, dx,$$

implies certain regularity of solutions to the divergence equation $\operatorname{div} \mathbf{u} = f$. Then by using duality, one gets the (weighted) Korn inequality. The proofs and results in [8] can be easily generalized to our setting, which enables us to deduce a (weighted) Korn inequality; see Section 2 below. For more on the recent progress on the divergence equation, see [3,4,8,18].

In Section 3 we discuss s-John domains and domains satisfying quasihyperbolic boundary conditions. By using Poincaré inequalities on these domains, we deduce the (weighted) Korn inequalities on them. Moreover, we will show the obtained (weighted) Korn inequalities are essentially sharp by presenting some counter-examples in Section 4.

The weighted Poincaré inequality on s-John domains is well known (see [12,20]), however, we could not find the results we needed for domains with quasihyperbolic boundary condition; for some related results see [10]. To this end, we will in Section 3 establish the weighted Poincaré inequality on such domains, which may have independent interest.

Another interesting question is what is the geometric counterpart of the Korn inequality. In general the Korn inequality (K_p) does not imply any Poincaré inequality. Indeed, if $\Omega_1, \Omega_2 \subset \mathbb{R}^n$, $\Omega_1 \cap \Omega_2 = \emptyset$, are two domains that support the Korn inequality (K_p), then $\Omega := \Omega_1 \cup \Omega_2$ admits the Korn inequality (K_p) as well. However, Poincaré inequality does not have this property.

In Section 2.2, we will show that, if the following Korn inequality

$$\int_\Omega |D\mathbf{v}|^p \, dx \leq C\left\{ \int_\Omega \left|\epsilon(\mathbf{v})\right|^p dx + \int_Q |\mathbf{v}|^p \, dx \right\} \qquad (\widetilde{K}_p)$$

holds for some cube $Q \subset\subset \Omega$, then there is a Poincaré inequality on Ω.

The paper is organized as follows. In Section 2, we will show that, abstractly, weighted Poincaré inequality implies a weighted Korn inequality; conversely, Korn inequality ($\widetilde{K}_p$) also implies a Poincaré inequality. In Section 3, we establish the Korn inequality on s-John domains and domains satisfying quasihyperbolic boundary conditions, and present examples for the sharpness of the Korn inequality in Section 4.

Throughout the paper, we denote by C positive constants which are independent of the main parameters, but which may vary from line to line. Corresponding to a function space X, we denote its n-vector valued spaces by X^n. We will usually omit the superscript n or $n \times n$ for simplicity. Let $Q(x, r) \subset \mathbb{R}^n$ denote the cube with the center x and side-length r, and for a constant $C > 0$, let CQ denote the cube $Q(x, Cr)$.

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