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A mixed boundary value problem for Chaplygin's hodograph equation

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ABSTRACT

In this paper we will prove existence, uniqueness and regularity of a classical solution to a mixed boundary value problem for Chaplygin's hodograph equation, which is degenerate elliptic on a part of the boundary. This problem is derived from the study of detached bow shock ahead of a straight ramp in uniform supersonic flows in the hodograph plane. The proof depends on Perron's method and some techniques from linear elliptic equations.

1. Introduction

It is well known that for a steady uniform supersonic flow past a straight ramp W', a detached bow shock $\mathcal{S}$ may appear ahead of it if the opening angle of the ramp is larger than a critical value (see Fig. 1 and *cf.* [22, p. 205, Sec. 4.12]). Rigorous analytical study of this problem is extremely difficult even if one assumes that the flow is isentropic and irrotational, *i.e.*, using the following potential flow equations[1]

$$v_x - u_y = 0, \tag{1.1a}$$

$$(\rho u)_x + (\rho v)_y = 0, \tag{1.1b}$$

where ρ is the density of mass, and (u, v) is the velocity of gas flow along the (x, y)-coordinates of the Euclidean plane. Up to now no simple special solution is available, since it involves nonlinear elliptic–hyperbolic

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[1] Note that in this paper we use subscript like u_x to denote the partial derivative $\frac{\partial u}{\partial x}$.

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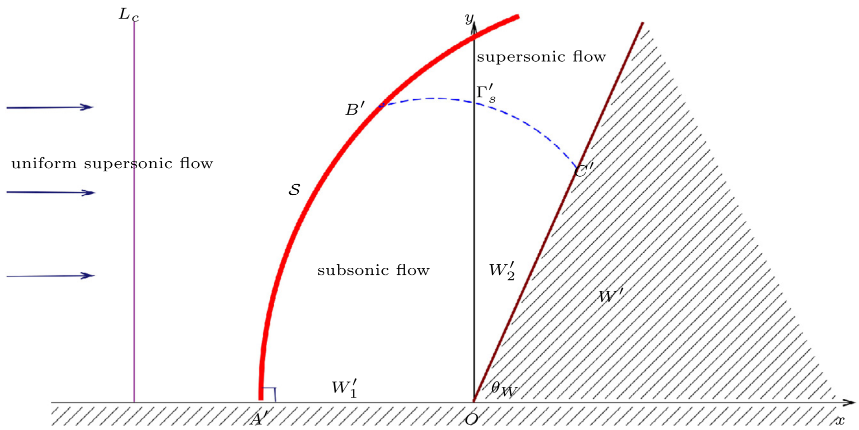

Fig. 1. The detached bow shock $\mathcal{S}$ ahead of a ramp W' with a large angle θ_W in uniform horizontal supersonic flows. Γ_s' is a sonic line separating subsonic flows in a neighborhood of O from supersonic downstream flows.

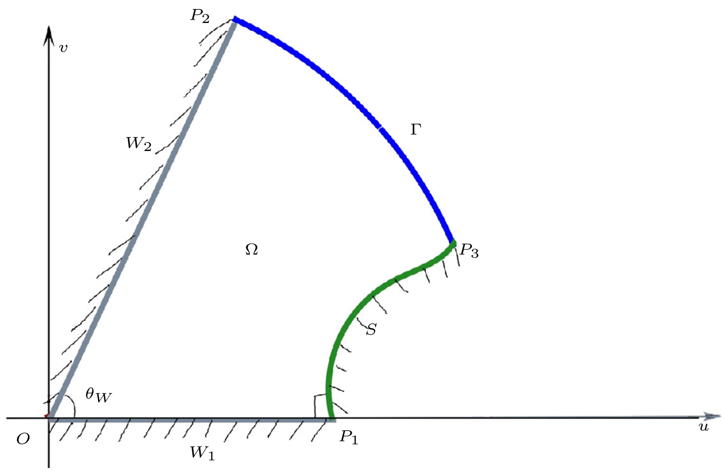

Fig. 2. The domain Ω.

mixed-type equations and free boundaries (transonic shocks). However, using the hodograph transformation, the nonlinear potential flow equations (1.1a)–(1.1b) can be transformed to Chaplygin's equation[2]

$$Q(\Phi) = \sum_{i,j=1}^{2} a^{ij}\partial_{ij}\Phi := \left(c^2 - v^2\right)\Phi_{uu} + 2uv\Phi_{uv} + \left(c^2 - u^2\right)\Phi_{vv} = 0, \tag{1.2}$$

where the function c (called the *sonic speed* in gas dynamics) is given by Bernoulli law [7, p. 23]

$$\mu^2\left(u^2 + v^2\right) + \left(1 - \mu^2\right)c^2 = c_*^2,$$

with the constants $c_* > 0$ and $\mu = \sqrt{(\gamma - 1)/(\gamma + 1)} \in (0, 1)$. Here $\gamma > 1$ is the adiabatic exponent for polytropic gas. The unknown shock-front (free boundary) $\mathcal{S}$ becomes the fixed boundary S given by the shock polar in the phase plane (u, v), and the surface of the ramp is transformed to W_2 and W_1 (see Fig. 2). The price is that the boundary condition on S becomes $\Phi_u - \Upsilon(\Phi_v) = 0$, where Υ is an unknown function such that $\Upsilon'(y) = v/(u_0 - u)$. In other words, $x = \Upsilon(y)$ is the equation of the shock-front in the physical plane (x, y). One can check that it is a non-classical nonlinear nonlocal oblique derivative condition.

[2] For details, see [7, p. 248, Sec. 103]. The unknown function $\Phi = \Phi(u, v)$ is introduced such that $\Phi_u = x$, $\Phi_v = y$.