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Remarks on multiplicity of solutions for a subquadratic elliptic equation

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ABSTRACT

We consider a subquadratic elliptic equation in a bounded domain $\Omega \subset \mathbb{R}^N$ ($N \geqslant 1$). By the Clark Theorem, we obtain the existence and multiplicity of its nontrivial solutions, and we show that this result has a great relationship with Ω itself. The above argument can be extended to biharmonic equations.

1. Introduction

Consider the boundary value problem for the elliptic equation

$$\begin{cases} -\Delta u(x) = V_u\big(x, u(x)\big), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{BVP}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geqslant 1$) is a bounded domain with a smooth boundary, the nonlinearity $V(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and $V_u(x,u) = \partial V/\partial u$ denotes the partial derivative of $V(x,u)$ with respect to u. If $V(x,u)$ satisfies $\limsup_{|u|\to\infty} V(x,u)/|u|^2 \leq a < \infty$, then we say that (BVP) is subquadratic. If $\lim_{|u|\to\infty} V(x,u)/|u|^2 = \infty$, then (BVP) is superquadratic. In this paper, our main goal is to find the existence and multiplicity of solutions of (BVP) for the subquadratic case.

Let λ_k ($k = 1, 2, \cdots$) denote the eigenvalues and ψ_k ($k = 1, 2, \cdots$) denote the corresponding eigenfunctions of the eigenvalue problem

$$\begin{cases} -\Delta u(x) = \lambda u(x), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where each λ_k is repeated as often as multiplicity such that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$. Recall that ψ_1 is positive (or negative) and the eigenfunctions associated with λ_k ($k \geqslant 2$) are sign-changing.

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In [7], Landsman and Lazer assumed that $V_u(x,u) = \lambda_1 u + g(x,u)$, where g is a bounded Carathéodory function for which $g_{\pm\infty}(x) = \lim_{s\to\pm\infty} g(x,s)$. If $\int_\Omega g_{+\infty}\psi_1 dx < 0 < \int_\Omega g_{-\infty}\psi_1 dx$ or $\int_\Omega g_{+\infty}\psi_1 dx > 0 > \int_\Omega g_{-\infty}\psi_1 dx$, then they have proved that (BVP) admits at least one solution. In [2], Brown studied (BVP) for $V_u(x,u) = \lambda u + b(x)|u|^{\gamma-2}u$, $1 < \gamma < 2$. He obtained, for every $\lambda < \lambda_1$, (BVP) has one solution if $\int_\Omega b\psi_1^\gamma dx > 0$; and whenever $\lambda_1 < \lambda \leqslant \lambda_1 + \delta$ for some small $\delta > 0$, (BVP) has two solutions if $\int_\Omega b\psi_1^\gamma dx < 0$. In [5], Costa and Willem assumed $V(x,u) = V(u)$ is strictly convex and $|V_u - \beta_1 u| \leqslant \beta_2|u| + \beta_3$ for some suitable $\beta_k > 0$ $(1 \leqslant k \leqslant 3)$, and they got a multiplicity result of (BVP). Motivated by [7,2,5] and references therein, in the present paper, we shall combine the subquadraticity of $V(x,u)$ with some special property of Ω to study (BVP). Exactly, we have the following result:

Theorem 1. *Let d_Ω denote the diameter of Ω. Suppose $V(x,u)$ satisfies that*

(V_1) $V(x,u) = V(x,-u)$, $\forall x \in \Omega$, $u \in \mathbb{R}$;
(V_2) *there exist* $m > 0$, $b > 0$ *such that* $d_\Omega < \frac{\pi}{\sqrt{m}}$, *and*

$$\left|V(x,u)\right| \le b + \frac{1}{2}m|u|^2, \quad \forall x \in \Omega,\ u \in \mathbb{R};$$

(V_3) *there exist* $p \in \mathbb{N}$, $M > 0$ *and* $\rho > 0$ *such that* $d_\Omega > \frac{2p\pi}{\sqrt{M}}$, $M > 4mp^2$, *and*

$$V(x,u) \ge \frac{1}{2}M|u|^2, \quad \forall x \in \Omega,\ |u| \le \rho\sqrt{p}.$$

Furthermore, if Ω satisfies that there exists an orthogonal function group $\{e_j(x)\}_{1\leqslant j\leqslant p} \subset \mathbb{C}(\overline{\Omega})$ in $H_0^1(\Omega)$ and $L^2(\Omega)$ respectively, such that

(V_4) $\int_\Omega |\nabla e_j(x)|^2 dx \le (\frac{2j\pi}{d_\Omega})^2 \int_\Omega |e_j(x)|^2 dx$, $\forall 1 \le j \le p$,

then, (BVP) has p distinct pairs $(u(x), -u(x))$ of nontrivial classical solutions, provided that $d_\Omega \in (\frac{2p\pi}{\sqrt{M}}, \frac{\pi}{\sqrt{m}})$.

Remark 1. If $V(x,u)$ satisfies

(V_2') $\lim_{|u|\to\infty} V(x,u)/|u|^2 = 0$ uniformly in $x \in \Omega$,
(V_3') $\lim_{|u|\to 0} V(x,u)/|u|^2 = \infty$ uniformly in $x \in \Omega$,

then, $\forall d_\Omega > 0$, $p \ge 1$, we can find $m > 0$, $M > 0$ and $\rho > 0$ such that (V_2)–(V_3) hold.

Remark 2. In [9], to some extent, we have discussed the case of $N = 1$ for Theorem 1, making use of Clark Theorem as follows.

Clark Theorem. *(See [3].) Let X be a Banach space and $\varphi \in \mathbb{C}^1(X,\mathbb{R})$ be even satisfying the Palais–Smale condition. Suppose that (i) φ is bounded from below; (ii) there exist a closed, symmetric set $K \subset X$ and $p \in \mathbb{N}$ such that K is homeomorphism to $S^{p-1} \subset \mathbb{R}^p$ by an odd map, and* $\sup\{\varphi(x) : x \in K\} < \varphi(0)$. *Then φ possesses at least p distinct pairs $(u,-u)$ of critical points with corresponding critical values less than $\varphi(0)$.*

For superquadratic case of (BVP), we refer the reader to [13,4,1].

The plan of the paper is as follows. In Section 2, by adapting some arguments employed in [9], we shall give the proof of Theorem 1. In Section 3, an application to $\Omega = B_r(0) \subset \mathbb{R}^2$ yields an interesting result: