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Solutions to the 2D Euler equations with velocity unbounded at infinity

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ABSTRACT

We prove existence of solutions to the two-dimensional Euler equations with vorticity bounded and with velocity locally bounded but growing at infinity at a rate slower than a power of the logarithmic function. We place no integrability conditions on the initial vorticity. This result improves upon a result of Serfati which gives existence of a solution to the two-dimensional Euler equations with bounded velocity and vorticity.

1. Introduction

We consider the Euler equations governing incompressible inviscid fluid flow in the plane, given by

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p \\ \operatorname{div} u = 0 \\ u|_{t=0} = u^0. \end{cases} \tag{E}$$

In this paper, we investigate the existence of solutions to (E) for which vorticity $\omega(u) = \partial_1 u_2 - \partial_2 u_1$ is bounded and velocity may grow at infinity, but at a rate slower than $\log_2^{1/4} |x|$.

The three-dimensional Euler equations with nondecaying velocity are considered in [6], where Constantin proves that there exists a solution to (E) with nondecaying velocity which blows up in finite time. In two dimensions, existence and uniqueness of solutions to (E) with $(u, \omega) \in L^\infty(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$ and without any integrability conditions on u or ω is established by Serfati [9]. Properties of Serfati solutions are further investigated in [1], where the authors extend the existence and uniqueness results of Serfati to an exterior domain. Building on results from [9], Taniuchi [10] proves existence of solutions to (E) with velocity bounded and vorticity belonging to the space Y^0_{ul}, which contains *bmo* and allows for unbounded vorticity (without placing any integrability conditions on u or ω). In [11], Taniuchi, Yoneda and Young establish uniqueness of solutions in a subset of Taniuchi's existence class by modifying methods from [12].

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Solutions to the two-dimensional Euler equations for which velocity grows at infinity are considered in [2]. The authors show that if initial velocity obeys the estimate

$$\left|u^0(x)\right| \le C\left(1+|x|^\alpha\right) \tag{1.1}$$

for some $\alpha \in [0,1)$ and the initial vorticity ω^0 belongs to $L^p \cap L^\infty(\mathbb{R}^2)$ for $p < 2/\alpha$, then the velocity satisfies (1.1) at all positive times. Similarly to [2], Brunelli [4] assumes that the initial velocity satisfies (1.1) with $\alpha = 1/2$ and that ω^0 is bounded and satisfies

$$\int_{\mathbb{R}^2} \frac{\omega^0(y)}{|x-y|}\,dy < \infty$$

for some $x \in \mathbb{R}^2$. He proceeds to show that under these assumptions, the growth rate of velocity is preserved at later times.

We remark that, in regard to the Navier–Stokes equations, short time existence of solutions in two and three dimensions with velocity bounded and nondecaying is shown in [7]. In [8], the authors show that in dimension two, the solution of [7] can be extended globally in time.

In this paper we consider solutions to (E) for which vorticity is bounded and potentially nondecaying and velocity may grow at infinity more slowly than a power of the logarithmic function. We prove the following theorem.

Theorem 1. *Let u^0 be such that gu^0 and $\omega^0 = \omega(u^0)$ belong to $L^\infty(\mathbb{R}^2)$, where $g(x) = \log_2^{-1/4}(2+|x|)$ for all $x \in \mathbb{R}^2$. There exists a weak solution u to (E) on $[0,\infty)$ with*

$$\begin{aligned} &gu \in L^\infty_{loc}\big([0,\infty), L^\infty\big(\mathbb{R}^2\big)\big), \quad \textit{and} \\ &\omega \in L^\infty\big([0,\infty), L^\infty\big(\mathbb{R}^2\big)\big). \end{aligned}$$

The proof of Theorem 1 consists of three steps: (i) Using the initial data u^0, we construct a sequence of smooth solutions (u_n) to the Euler equations which lie in our existence class and which converge uniformly on compact subsets of $\mathbb{R}^2$. (ii) We establish an upper bound on the L^∞-norms of the sequence (gu_n) which is independent of n. (iii) We pass to the limit and use the uniform bound from step (ii) to show that the limit u is a solution to (E) in our existence class with initial data u^0.

To establish the uniform bound in step (ii), we let u be a smooth solution to (E), fix $N \le -1$, and write gu as a sum of two terms:

$$gu(t,x) = gS_N u(t,x) + g(Id - S_N)u(t,x), \tag{1.2}$$

where $S_N u = \chi_N * u$, $\chi_N = 2^{2N}\chi(2^N\cdot)$, and χ is a radial Schwartz function which integrates to one. One can easily estimate the L^∞-norm of the second term of (1.2) using membership of ω to L^∞. The first term is more delicate. The main obstacle lies in estimating the pressure, specifically terms of the form

$$g(x) \sum_{i,j=1,2} \nabla S_N R_i R_j (u_i u_j), \tag{1.3}$$

where R_k denotes the Riesz operator. Since the Riesz operators are not bounded on $L^\infty(\mathbb{R}^2)$, we write (1.3) as a convolution and apply the Riesz operators to the function $\nabla\chi_N$. We are then able to bound (1.3) by

$$g(x)\|gu\|^2_{L^\infty} \int_{\mathbb{R}^2} \left|R_i R_j \nabla\chi_N(y)\right| \left(1/g^2\right)(x-y)\,dy, \tag{1.4}$$