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Overdetermined partial boundary value problems on finite networks

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ABSTRACT

In this study, we define a class of non-self-adjoint boundary value problems on finite networks associated with Schrödinger operators. The novel feature of this study is that no data are prescribed on part of the boundary, whereas both the values of the function and of its normal derivative are given on another part of the boundary. We show that overdetermined partial boundary value problems are crucial for solving inverse boundary value problems on finite networks since they provide the theoretical foundations for the recovery algorithm. We analyze the uniqueness and the existence of solution for overdetermined partial boundary value problems based on the nonsingularity of partial Dirichlet-to-Neumann maps. These maps allow us to determine the value of the solution in the part of the boundary where no data was prescribed. We also execute full conductance recovery for spider networks.

1. Introduction

The *inverse boundary value problem* was described for the first time around 1950 by A. Calderón. However, it was not until 1980 that he published "On an inverse boundary value problem" [10] on this subject. This problem appears to be a consequence of an engineering problem on geophysical electrical prospection where the objective is to deduce some internal terrain properties from surface electrical measurements.

These studies have motivated several developments in the inverse problem field until the present day. More recently, this problem has also been considered for medical applications in *electrical impedance tomography* (EIT) [11], which is a medical imaging technique where an image that contains visual information related to internal body parts is obtained from electrical measurements on the boundary.

The corresponding mathematical problem proposed by Calderón is whether it is possible to determine the conductivity of a body based on current and voltage measurements at its boundary. This problem of recovering conductances from boundary or surface current and potential measurements is a nonlinear inverse problem, which is exponentially ill-posed [1,17] since its solution is highly sensitive to changes in the boundary data.

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Since its appearance, Calderón's inverse problems have been treated in many ways. For instance, in [9, 18], Sylvester and Uhlmann treated the uniqueness of the solution. Curtis, Ingerman, and Morrow worked on critical circular planar networks conductivity reconstruction [12–14,16]. Borcea, Druskin, Guevara, and Mamonov have analyzed EIT problems in depth and their most recent study treated numerical conductivity reconstruction [6,8,7].

Inverse boundary value problems have been considered over both the continuum and discrete fields. In this study, we define a new class of boundary value problems on finite networks associated with Schrödinger operators. The novel feature of this study is that no data are prescribed on a part of the boundary, whereas both the values of the function and its normal derivative are given on another part of the boundary. These problems are not self-adjoint, thus we are concerned with the study of existence and uniqueness through the adjoint problem.

We show that overdetermined partial boundary value problems are crucial in the framework of inverse boundary value problems on finite networks since they provide the theoretical foundations of the recovery algorithm. In fact, this type of problem was considered implicitly in some previous studies, but only for specific networks and boundary data (see [13,14]). We analyze the uniqueness and existence of the solution to overdetermined partial boundary value problems through the nonsingularity of partial Dirichlet-to-Neumann maps. These maps allow us to determine the value of the solution on part of the boundary with no prescribed data. Next, we give explicit formulae for obtaining boundary spike conductances on critical planar networks and we execute full conductance recovery for spider networks. This algorithm is an adaptation of that proposed in [14] for the Combinatorial Laplacian when the corresponding Dirichlet-to-Neumann map is singular.

2. Preliminaries

Let $\Gamma = (V, c)$ be a finite network, i.e., a finite connected graph without loops or multiple edges, and with the vertex set V. Let E be the set of edges of the network Γ. Each edge (x, y) is assigned a *conductance* $c(x, y)$, where $c : V \times V \longrightarrow [0, +\infty)$. Moreover, $c(x, y) = c(y, x)$ and $c(x, y) = 0$ if $(x, y) \notin E$. Then, $x, y \in V$ are *adjacent*, $x \sim y$, iff $c(x, y) > 0$.

The set of functions on a subset $F \subseteq V$, denoted by $\mathcal{C}(F)$, and the set of nonnegative functions on F, $\mathcal{C}^+(F)$, are identified naturally with $\mathbb{R}^{|F|}$ and the nonnegative cone of $\mathbb{R}^{|F|}$, respectively. We denote $\int_F u(x)dx$ by the value $\sum_{x\in F} u(x)$. Moreover, if F is a nonempty subset of V, its characteristic function is denoted by χ_F. When $F = \{x\}$, its characteristic function is denoted by ε_x. If $u \in \mathcal{C}(V)$, we define the *support* of u as $\mathsf{supp}(u) = \{x \in V : u(x) \neq 0\}$.

If we consider a proper subset $F \subset V$, then its *boundary* $\delta(F)$ is given by the vertices of $V \setminus F$ that are adjacent to at least one vertex of F. The vertices of $\delta(F)$ are called *boundary vertices* and when a boundary vertex $x \in \delta(F)$ has a unique neighbor in F, we refer to the edge joining them as a *boundary spike*. It is easy to prove that $\bar{F} = F \cup \delta(F)$ is connected when F is connected. Any function $\omega \in \mathcal{C}^+(\bar{F})$ such that $\mathsf{supp}(\omega) = \bar{F}$ and $\int_{\bar{F}} \omega^2(x)\,dx = 1$ is called a *weight* on $\bar{F}$. The set of weights is denoted by $\Omega(\bar{F})$. We denote $\kappa_F \in \mathcal{C}(\delta(F))$ as the function $\kappa_F(x) = \sum_{y\in F} c(x, y)$.

We define the *normal derivative of* $u \in \mathcal{C}(\bar{F})$ *on* F as the function in $\mathcal{C}(\delta(F))$ given by

$$\left(\frac{\partial u}{\partial \mathsf{n}_F}\right)(x) = \int_F c(x, y)\big(u(x) - u(y)\big)\,dy, \quad \text{for any } x \in \delta(F).$$

Any function $K \in \mathcal{C}(F \times F)$ is called a *kernel on* F. The *integral operator associated with* K is the endomorphism $\mathscr{K}:\mathcal{C}(F) \longrightarrow \mathcal{C}(F)$, which assigns to each $f \in \mathcal{C}(F)$, the function $\mathscr{K}(f)(x) = \int_F K(x, y)\, f(y)\,dy$ for all $x \in V$. Conversely, given an endomorphism $\mathscr{K}:\mathcal{C}(F) \longrightarrow \mathcal{C}(F)$, the associated kernel is given by