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Nonlocal diffusion, a Mittag-Leffler function and a two-dimensional Volterra integral equation

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ABSTRACT

In this paper we consider a particular class of two-dimensional singular Volterra integral equations. Firstly we show that these integral equations can indeed arise in practice by considering a diffusion problem with an output flux which is nonlocal in time; this problem is shown to admit an analytic solution in the form of an integral. More crucially, the problem can be re-characterized as an integral equation of this particular class. This example then provides motivation for a more general study: an analytic solution is obtained for the case when the kernel and the forcing function are both unity. This analytic solution, in the form of a series solution, is a variant of the Mittag-Leffler function. As a consequence it is an entire function. A Gronwall lemma is obtained. This then permits a general existence and uniqueness theorem to be proved.

1. Introduction

In this article we shall consider the class of second kind Volterra integral equations of the form

$$y(t) = \int_0^t \int_0^\tau \frac{k(t,\tau,\sigma)y(\sigma)}{(t-\tau)^\alpha(\tau-\sigma)^\beta}\, d\sigma\, d\tau + f(t), \quad 0 \le \alpha,\ \beta < 1, \tag{1.1}$$

where $(t,\tau,\sigma) \in \Omega \doteq \{0 \le \sigma \le \tau \le t \le T\}$ and $y(0) = f(0)$.

In addition, the functions k and f are assumed to be sufficiently smooth and $k(t,t,t) \not\equiv 0$ for all $t \in [0,T]$.

2. A diffusion problem

Consider the nonlocal (in time) diffusion problem

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$$\frac{\partial c}{\partial t}(x,t) = \frac{\partial^2 c}{\partial x^2}(x,t), \quad 0 < x < 1,\ t > 0, \tag{2.1}$$

$$\frac{\partial c}{\partial x}(0,t) = 0, \quad t > 0, \tag{2.2}$$

$$\frac{\partial c}{\partial x}(1,t) = -\int_0^t \frac{1}{\sqrt{\pi(t-\tau)}} c(1,\tau)\, d\tau, \quad t > 0, \tag{2.3}$$

subject to initial condition $c(x,0) = c_0$.

Take Laplace Transforms with respect to t:

$$\frac{d^2\bar{c}}{dx^2}(x,s) - s\bar{c}(x,s) = -c_0$$

yielding the solution

$$\bar{c}(x,s) = A(s)\cosh\sqrt{s}x + B(s)\sinh\sqrt{s}x + \frac{c_0}{s},$$

where $\bar{c}(x,s) = \int_0^\infty e^{-st}c(x,t)dt$.

From (2.2) we have

$$\bar{c}(x,s) = A(s)\cosh\sqrt{s}x + \frac{c_0}{s}.$$

Set $x = 1$ and solve for $A(s)$:

$$A(s) = \left(\bar{c}(1,s) - \frac{c_0}{s}\right)/\cosh\sqrt{s}$$

or

$$\bar{c}(x,s) = \frac{(\bar{c}(1,s) - \frac{c_0}{s})}{\cosh\sqrt{s}}\cosh\sqrt{s}x + \frac{c_0}{s}. \tag{2.4}$$

Differentiate with respect to x and employ (2.3):

$$\frac{(\bar{c}(1,s) - \frac{c_0}{s})}{\cosh\sqrt{s}}\sqrt{s}\sinh\sqrt{s} = -L\left[\int_0^t \frac{1}{\sqrt{\pi(t-\tau)}} c(1,\tau)\, d\tau\right] = -\frac{1}{\sqrt{s}}\bar{c}(1,s)$$

by convolution.

Therefore

$$\bar{c}(1,s) = \frac{c_0}{s} - \left(\frac{\coth\sqrt{s}}{\sqrt{s}}\right)\left(\frac{1}{\sqrt{s}}\right)\bar{c}(1,s). \tag{2.5}$$

However [2],

$$L^{-1}\left[\frac{\coth\sqrt{s}}{\sqrt{s}}\right] = 1 + 2\sum_{n=1}^\infty e^{-n^2\pi^2 t} = \frac{1}{\sqrt{\pi t}}\left(1 + 2\sum_{n=1}^\infty e^{-n^2/t}\right). \tag{2.6}$$

Using (2.6) and applying convolution twice we observe that (2.5) transforms to

$$c(1,t) = c_0 - \int_0^t\int_0^\tau \frac{1}{\sqrt{\pi(t-\tau)}}\left(1 + 2\sum_{n=1}^\infty e^{-n^2/(t-\tau)}\right)\frac{1}{\sqrt{\tau-\sigma}}c(1,\sigma)\, d\sigma\, d\tau$$