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Interpolating solutions of the Poisson equation in the disk based on Radon projections

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ABSTRACT

We consider an algebraic method for reconstruction of a function satisfying the Poisson equation with a polynomial right-hand side in the unit disk. The given data, besides the right-hand side, is assumed to be in the form of a finite number of values of Radon projections of the unknown function. We first homogenize the problem by finding a polynomial which satisfies the given Poisson equation. This leads to an interpolation problem for a harmonic function, which we solve in the space of harmonic polynomials using a previously established method. For the special case where the Radon projections are taken along chords that form a regular convex polygon, we extend the error estimates from the harmonic case to this Poisson problem. Finally we give some numerical examples.

1. Introduction

The classical approach to interpolation is based on sampling a given function at a finite number of points. This is natural for approximation of univariate functions since a table of function values is a standard type of information in practical problems and processes described by functions in one variable. Moreover, the Lagrange interpolation problem by polynomials is always uniquely solvable.

In the multivariate case, such an approach is met with serious difficulties. For example, the pointwise interpolation by multivariate polynomials is no longer possible for every choice of the nodes. See [6] and the references therein for a survey of multivariate polynomial interpolation. Furthermore, there are many practical problems in which the information about the relevant function comes as a set of functionals which are not point evaluations. For instance, in computer tomography, a table of mean values of a function of d variables on $(d-1)$-dimensional hyperplanes is the data on which the reconstruction is based. Such

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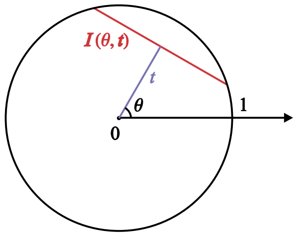

Fig. 1. The chord $I(\theta, t)$ of the unit circle.

nondestructive methods have important practical applications in medicine, radiology, geology, etc., and have their theoretical foundation in the work of Johann Radon in the early twentieth century [22].

Mathematically speaking, the problem is to recover or approximate a multivariate function using information given as integrals of the unknown function over a number of hyperplanes. This problem has been intensively studied since the 1960s using different approaches [3–5,16–21,23] and continues to find many applications. Among the developed reconstruction algorithms are filtered backprojection, iterative reconstruction, direct methods, etc., and some are based on the inverse Radon transform.

One class of methods uses direct interpolation by multivariate polynomials [1,2,10–15,20]. In our work, we follow this general approach. The interpolant is sought in an appropriate polynomial space such that it matches the given Radon projections exactly.

To improve the approximation accuracy and to reduce the amount of input data required as well as the computational effort, it seems natural to incorporate additional knowledge about the function to be recovered into the approximation method. This was first suggested by Borislav Bojanov. Such problem-specific knowledge is often provided in the form of a partial differential equation which the unknown function satisfies.

In the present paper, we concern ourselves with the case where the unknown function satisfies the Poisson equation $\Delta u = u_{xx} + u_{yy} = f$, where f is a polynomial. This elliptic partial differential equation is important both as a model problem as well as in applications.

The present work expands on the earlier articles [7,9], where the Laplace equation was considered, i.e., the homogeneous case $f = 0$. Therein, first results on interpolation of harmonic functions with harmonic polynomials based on Radon projections along chords of the unit circle were presented. The existence of a unique interpolant in the space of harmonic polynomials was shown for a family of schemes where all chords are chosen at equal distance to the origin. For the special case of chords forming a regular convex polygon, error estimates on the unit circle and in the unit disk were proved.

In the present paper, our main aim is to extend several of these results to the inhomogeneous case, i.e., the Poisson equation, with a polynomial right-hand side, again using Radon projections type of data. Both the Laplace and the Poisson equation have many practical applications such as heat transport, diffusion problems or in Stokes flow of incompressible fluids, making them interesting both as model problems and with a view to applications. The main idea is to reduce the interpolation problem to the harmonic case by finding a suitable homogenizing polynomial. This allows us to prove existence and uniqueness of a polynomial interpolant. We obtain an error estimate under certain more restrictive assumptions.

2. Preliminaries

Let $D \subset \mathbb{R}^2$ denote the open unit disk and ∂D the unit circle. By $I(\theta, t)$ we denote a chord of the unit circle at angle $\theta \in [0, 2\pi)$ and distance $t \in (-1, 1)$ from the origin (see Fig. 1), parameterized by

$$s \mapsto (t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)^\top, \quad \text{where } s \in \left(-\sqrt{1-t^2}, \sqrt{1-t^2}\right).$$