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A class of spectral self-affine measures with four-element digit sets

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A R T I C L E I N F O

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A B S T R A C T

The self-affine measure $\mu_{M,D}$ associated with an expanding matrix $M \in M_n(\mathbb{Z})$ and a finite digit set $D \subset \mathbb{Z}^n$ is uniquely determined by the self-affine identity with equal weight. In this paper we construct a class of self-affine measures $\mu_{M,D}$ with four-element digit sets in the higher dimensions ($n \geq 3$) such that the Hilbert space $L^2(\mu_{M,D})$ possesses an orthogonal exponential basis. That is, $\mu_{M,D}$ is spectral. Such a spectral measure cannot be obtained from the condition of compatible pair. This extends the corresponding result in the plane.

1. Introduction and notations

For a probability measure μ of compact support on $\mathbb{R}^n$, we call μ a *spectral measure* if there exists a discrete set $\Lambda \subset \mathbb{R}^n$ such that $E(\Lambda) := \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ forms an orthogonal basis (Fourier basis) for $L^2(\mu)$. The set Λ is then called a *spectrum* for μ; we also say that (μ, Λ) is a *spectral pair*. It is known that (μ, Λ) is a spectral pair if and only if

$$\sum_{\lambda\in\Lambda}\left|\hat{\mu}(\xi+\lambda)\right|^2 = 1 \quad \left(\forall \xi \in \mathbb{R}^n\right), \tag{1.1}$$

where $\hat{\mu}$ denotes the Fourier transform of μ. In this paper we determine the spectrality of a class of self-affine measures with four-element digit sets in the higher dimensions ($n \geq 3$). The self-affine measure considered here is the unique probability measure $\mu := \mu_{M,D}$ satisfying the self-affine identity with equal weight:

$$\mu = \frac{1}{|D|}\sum_{d\in D}\mu\circ\phi_d^{-1}, \tag{1.2}$$

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where $\phi_d(x) = M^{-1}(x+d)$ is an affine mapping, $M \in M_n(\mathbb{Z})$ is an expanding integer matrix, and $D \subset \mathbb{Z}^n$ is a finite digit set of cardinality $|D|$. Such a measure $\mu_{M,D}$ is supported on the *attractor* or *invariant set* $T(M,D)$ of the affine iterated function system (IFS) $\{\phi_d(x)\}_{d\in D}$. This is the unique nonempty compact subset $T := T(M,D) \subset \mathbb{R}^n$ satisfying $T = \bigcup_{d\in D} \phi_d(T)$ (see [5]). From (1.2), the Fourier transform $\hat{\mu}_{M,D}(\xi)$ of the measure $\mu_{M,D}$ is given by

$$\hat{\mu}_{M,D}(\xi) = \int e^{2\pi i\langle x,\xi\rangle} d\mu_{M,D}(x) = \prod_{j=1}^{\infty} m_D\big(M^{*-j}\xi\big) \quad \big(\xi \in \mathbb{R}^n\big), \tag{1.3}$$

where M^* denotes the transposed conjugate of M (in fact, $M^* = M^t$) and

$$m_D(x) = \frac{1}{|D|}\sum_{d\in D} e^{2\pi i\langle d,x\rangle} \quad \big(x \in \mathbb{R}^n\big). \tag{1.4}$$

For a given pair (M,D), the spectrality of $\mu_{M,D}$ is directly connected with the above (1.1) and (1.3), where the function (1.4) plays an important role.

Let $S \subset \mathbb{Z}^n$ be a finite subset of the cardinality $|S| = |D|$ and $0 \in S$. Corresponding to the dual IFS $\{\psi_s(x) = M^*x + s\}_{s\in S}$, we use $\Lambda(M,S)$ to denote the expansive orbit of 0 under $\{\psi_s(x)\}_{s\in S}$, that is,

$$\Lambda(M,S) := \left\{ \sum_{j=0}^{k} M^{*j} s_j : k \geq 0 \text{ and all } s_j \in S \right\}.$$

It was first observed by Jorgensen and Pedersen [6] that for certain M, D and S, the corresponding $\mu_{M,D}$ is a spectral measure with spectrum $\Lambda(M,S)$. This is surprising because some classical results on Fourier analysis can be established on the fractal sets. Later, Strichartz [15–17] extended the construction of [6] to a large class of measures. In all such researches, the condition that $(M^{-1}D,S)$ is a compatible pair (or (M,D,S) is a Hadamard triple) plays an essential role (see [2, Conjecture 2.5], [3, Problem 1] and [4, Conjecture 1.1]).

Definition 1.1. For two finite subsets G and P of $\mathbb{R}^n$ of the same cardinality q, we say (G,P) is a compatible pair if the $q \times q$ matrix

$$H_{G,P} := \big[q^{-1/2} e^{2\pi i\langle g,p\rangle}\big]_{g\in G,\, p\in P}$$

is unitary, i.e. $H_{G,P}H^*_{G,P} = I_q$.

The well-known result of Jorgensen and Pedersen [6] shows that if $(M^{-1}D,S)$ is a compatible pair with the expanding matrix $M \in M_n(\mathbb{Z})$ and $D,S \subset \mathbb{Z}^n$, then $E(\Lambda(M,S))$ is an infinite orthogonal system in $L^2(\mu_{M,D})$. Łaba and Wang [7] extended this result in the dimension one ($n = 1$), and showed that $\mu_{M,D}$ is always a spectral measure. In the higher dimensions ($n \geq 2$), it often needs additional condition for $E(\Lambda(M,S))$ to be an orthogonal basis in $L^2(\mu_{M,D})$ (see [12, Question 1.1]). In the plane ($n = 2$), there are several methods to deal with the $\mu_{M,D}$-orthogonal exponentials in the case when $|D| = 2,3,4$ (see [13] and references cited there). Most of them are suitable to the case $n \geq 3$. However, for the four-element digit set in the space ($n = 3$), the spectrality or the non-spectrality of $\mu_{M,D}$ is discussed only in the case $M = \mathrm{diag}(p_1,p_2,p_3)$ and $D = \{0,e_1,e_2,e_3\}$ (see [11]), where e_1, e_2, e_3 are the standard basis of unit column vectors in $\mathbb{R}^3$. Moreover, the spectral measure is obtained only in the special case when $p_1,p_2,p_3 \in 2\mathbb{Z}\setminus\{0,2\}$ or when $p_1 = p_2 = p_3 = p$, $p \in 2\mathbb{Z} \setminus \{0\}$. It should be pointed out that the results on such spectral measures provide some supportive evidence on the conjecture that if $(M^{-1}D,S)$ is a compatible pair with