J. Math. Anal. Appl. 423 (2015) 336–357

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On progressive functions

Anca Deliu [a], Baiqiao Deng [b,*]

[a] *Department of Mathematics, Georgia Perimeter College, Clarkston, GA 30021, USA*
[b] *Department of Mathematics, Columbus State University, Columbus, GA 31907, USA*

ARTICLE INFO

Article history:
Received 9 May 2014
Available online 8 October 2014
Submitted by P.G. Lemarie-Rieusset

Keywords:
Progressive functions
Shift-invariant spaces
Orthonormal sequence
Riesz sequence
Dual functions

ABSTRACT

Progressive functions at time t involve only the progressive functions at time before t and some nice compactly supported function at time t. We give sufficient conditions and explicit formulas to construct progressive functions with exponential decay and characterize the conditions on which the positive integer translates of a progressive function are orthonormal or a Riesz sequence. We provide explicit ways for construction of orthonormal progressive functions and for construction of the biorthogonal functions of nonorthogonal progressive functions. Such progressive functions can be used to construct wavelets with arbitrary smoothness on the half line if they are generated by a smooth refinable compactly supported function.

1. Introduction

In this paper we investigate real valued functions $\phi(x)$ as solutions of the functional equation:

$$\phi(x) = Cs(x) + \sum_{k=1}^{N-1} \alpha_k \phi(x-k), \tag{1.1}$$

where $s \in L_2(\mathbf{R})$ is a real valued function with support in $[0,\infty)$, $\alpha_1, \alpha_2, \ldots, \alpha_{N-1}$ and C are real numbers with $C \neq 0$, and N is an integer such that $N \geq 2$. A function $g \in L_2(\mathbf{R})$ is called progressive if and only if its Fourier transform is supported in $[0,\infty)$. The space of progressive functions is a closed subspace of $L_2(\mathbf{R})$; in fact it is the Hardy space $H^2(\mathbf{R})$. These functions play an important role in continuous wavelet analysis and signal processing cf. [13,15]. Functions whose Fourier transform is supported in $[0,\infty)$, or in the periodic case whose Fourier coefficients are supported in the set of non-negative integers, are interpreted in physics as causal solutions, that is, those solutions that govern processes in which the future cannot affect the past. A solution $\phi(x)$ of (1.1) turns out to be a series of non-negative integer translates of the function $s(x)$ (see (3.2)). Such a series reflects the process through which live video transmission is implemented. If we view

* Corresponding author. Fax: +1 706 507 8263.
E-mail addresses: Anca.Deliu@gpc.edu (A. Deliu), bdeng@ColumbusState.edu (B. Deng).

http://dx.doi.org/10.1016/j.jmaa.2014.10.001

$\phi(x)$ and $s(x)$ as images at the current time and $s(x-n)$ as an image at a previous time n if $n>0$, then $\phi(x)$ can be progressively recovered from the images $s(x-n)$ at the current and previous times. In other words, the present can be recovered from the past and is not affected by the future and thus by analogy we call a solution $\phi(x)$ of (1.1) a *progressive function*. Since the function $\phi(x)$ is completely determined by $s(x)$, a function located at the present time on the time line, we call $s(x)$ the *updating function*.

In a more general sense a progressive function is the solution $\phi(x)$ of the following functional equation:

$$\phi(x) = Cs(x) + \sum_{k=1}^{\infty} \alpha_k \phi(x-k), \tag{1.2}$$

for an updating function $s \in L_2(\mathbf{R})$ whose support is in $[0,\infty)$ and infinite sequence of real numbers $\{\alpha_k\}_{k=1}^{\infty} \in \ell_1(\mathbb{N})$. Equivalently, $\phi(x)$ is a solution of the following functional equation:

$$s(x) = \sum_{k=0}^{\infty} \alpha_k \phi(x-k), \tag{1.3}$$

where $\{\alpha_k\}_{k=0}^{\infty} \in \ell_1(\mathbb{N}_0)$ and $\alpha_0 \neq 0$. As usual we use $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}$ and $\mathbb{R}$ to denote the set of positive integers, nonnegative integers, integers, and real numbers, respectively. Applying Fourier transform on both sides of (1.3), we have

$$\widehat{s}(\xi) = p\bigl(e^{-i\xi}\bigr)\widehat{\phi}(\xi), \tag{1.4}$$

where $p(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ is a Laurent series. A Laurent series is said to belong to the Wiener class W if its coefficient sequence is in $\ell_1(\mathbb{Z})$. The well known Wiener Lemma states that if $f \in W$ and $f \neq 0$ for all z on the unit circle $|z|=1$, then $\frac{1}{f} \in W$ (see [12,18,21,23]). Thus, if $p(z) \neq 0$ for all z on the unit circle $|z|=1$, then we can define $D(z) = 1/p(z) \in W$ and we have

$$\widehat{\phi}(\xi) = D\bigl(e^{-i\xi}\bigr)\widehat{s}(\xi) = \sum_k d_k e^{-ik\xi}\widehat{s}(\xi), \tag{1.5}$$

for some sequence $\{d_k\} \in \ell_1(\mathbb{Z})$. Thus a progressive function can be written as

$$\phi(x) = \sum_k d_k s(x-k). \tag{1.6}$$

This indicates that progressive functions have a close connection to principal shift-invariant spaces. A closed subspace $V \subset L_2(\mathbb{R})$ is shift-invariant if $f \in V$ implies that $f(x-k) \in V$ for all $k \in \mathbb{Z}$. For a subset Φ of $L_2(\mathbb{R})$, the shift-invariant space generated by Φ is denoted by $S(\Phi)$ and is the smallest closed shift invariant space that contains Φ. If Φ is a finite subset of $L_2(\mathbb{R})$, then $S(\Phi)$ is said to be finitely generated shift-invariant space. If Φ consists of a single function φ, then $S(\varphi)$ is called a principal shift-invariant space generated by φ. For a complete characterization of Riesz bases, linear independence, frames, and approximation order in shift-invariant spaces one can refer to [1,8,9,16,17] and the references therein. Principal shift-invariant spaces with a refinable generator also played an important role in construction of wavelets using multiresolution analysis [2,5,6]. It is easy to see from (1.6) that if $s(x)$ is a refinable function, then $\phi(x)$ is also a refinable function. In this case, a multiresolution analysis on the positive half line can be generated by $\phi(x)$ if the positive integer translates of $\phi(x)$ form a Riesz sequence and therefore orthogonal or biorthogonal wavelets on the positive half line can be constructed. With the Walsh–Fourier transform, compactly supported wavelets on the positive half line were extensively studied [11,20]. The progressive functions ϕ satisfying (1.1) can be used to construct wavelets with arbitrary smoothness on the positive half line if they are generated by