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Uniqueness of positive radial solutions for a class of semipositone problems on the exterior of a ball

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ABSTRACT

We study positive radial solutions to: $-\Delta u = \lambda K(|x|)f(u)$; $x \in \Omega_e$, where $\lambda > 0$ is a parameter, $\Omega_e = \{x \in \mathbb{R}^N : |x| > r_0,\ r_0 > 0,\ N > 2\}$, Δ is the Laplacian operator, $K \in C([r_0,\infty),(0,\infty))$ satisfies $K(r) \leq \frac{1}{r^{N+\mu}}$; $\mu > 0$ for $r \gg 1$ and $f \in C^1([0,\infty),\mathbb{R})$ is a concave increasing function satisfying $\lim_{s\to\infty} \frac{f(s)}{s} = 0$ and $f(0) < 0$ (semipositone). We are interested in solutions u such that $u \to 0$ as $|x| \to \infty$ and satisfy the nonlinear boundary condition $\frac{\partial u}{\partial \eta} + \tilde{c}(u)u = 0$ if $|x| = r_0$ where $\frac{\partial}{\partial \eta}$ is the outward normal derivative and $\tilde{c} \in C([0,\infty),(0,\infty))$ is an increasing function. We will establish the uniqueness of positive radial solutions for large values of the parameter λ.

1. Introduction

We consider the steady state reaction diffusion equation on the exterior of a ball in $\mathbb{R}^N$; $N > 2$ with nonlinear boundary conditions on the interior boundary. Namely, we study positive radial solutions to:

$$\begin{cases} -\Delta u = \lambda K\big(|x|\big)f(u), & x \in \Omega_e, \\ \dfrac{\partial u}{\partial \eta} + \tilde{c}(u)u = 0, & \text{if } |x| = r_0 \\ u \to 0, & \text{if } |x| \to \infty, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $\Omega_e = \{x \in \mathbb{R}^N : |x| > r_0,\ r_0 > 0,\ N > 2\}$, Δ is the Laplacian operator, $K \in C([r_0,\infty),(0,\infty))$ satisfies $K(r) \leq \frac{1}{r^{N+\mu}}$; $\mu > 0$ for $r \gg 1$, $\frac{\partial}{\partial \eta}$ is the outward normal derivative and $\tilde{c} \in C([0,\infty),(0,\infty))$ is an increasing function. Here $f : [0,\infty) \to \mathbb{R}$ is a C^1 function. The case when $f(0) < 0$ is referred to in the literature as semipositone problems and has been well documented (see [4,14]) that the

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study of positive solutions to such problems is considerably more challenging than in the case $f(0) > 0$ (positone problems). For a rich history on semipositone problems with Dirichlet boundary conditions on bounded domains, see [1–3,6,7,9–11,15] and on domains exterior to a ball, see [8,13,16]. Such nonlinear boundary conditions occur very naturally in applications (see [12] for a detailed description in a model arising in combustion theory). Recently, the existence of a radial positive solution for (1.1) when $\lambda \gg 1$ has been established in [5], via the method of sub–super solutions. Here we discuss the uniqueness of this radial solution when some additional assumptions hold.

In [8], the authors study such a uniqueness result for the case of Dirichlet boundary condition on $|x| = r_0$. Our focus in this paper is to consider the uniqueness result for semipositone problem when a class of nonlinear boundary condition is satisfied at $|x| = r_0$. The fact that we have no longer a fixed value of u on $|x| = r_0$ results in quite a challenge in extending the results in [8]. Namely, we need to establish a detailed behavior of u at $|x| = r_0$ to achieve our goal.

Note that the change of variable $r = |x|$ and $s = (\frac{r}{r_0})^{2-N}$ transforms (1.1) into the following boundary value problem (see Appendix 8.1 in [5] for details):

$$\begin{cases} -u''(t) = \lambda \tilde{h}(t) f\big(u(t)\big), \quad t \in (0,1), \\ \dfrac{(N-2)}{r_0} u'(1) + \tilde{c}\big(u(1)\big)u(1) = 0, \\ u(0) = 0, \end{cases} \tag{1.2}$$

where $\tilde{h}(t) = \frac{r_0^2}{(2-N)^2} t^{\frac{-2(N-1)}{N-2}} K(r_0 t^{\frac{1}{2-N}})$. We will only assume $K(r) \leq \frac{1}{r^{N+\mu}}$ for $r \gg 1$ and for some $\mu \in (0, N-2)$. Then, $\tilde{h} \in C^1((0,1],(0,\infty))$ could be singular at 0. If $\mu \geq N-2$, $\tilde{h}$ will be nonsingular at 0 and it will be an easier case to study. Note that $\inf_{t\in(0,1]} \tilde{h}(t) > 0$, and there exists a constant $\tilde{d}$ such that $\tilde{h}(t) \leq \frac{\tilde{d}}{t^{\tilde{\alpha}}}$ for all $t \in (0, \tilde{\epsilon})$ where $\tilde{\alpha} = \frac{N-2-\mu}{N-2}$ and $\tilde{\epsilon} \approx 0$.

In view of the above discussion, we study positive solutions in $C^2(0,1) \cap C^1[0,1]$ to the following boundary problem:

$$\begin{cases} -u''(t) = \lambda h(t) f\big(u(t)\big), \quad t \in (0,1), \\ u'(1) + c\big(u(1)\big)u(1) = 0, \\ u(0) = 0, \end{cases} \tag{1.3}$$

where $c \in C([0,\infty),(0,\infty))$ is an increasing function and $h \in C^1((0,1],(0,\infty))$ satisfying:

(H1) $\underline{h} := \inf_{t\in(0,1]} h(t) > 0$;
(H2) there exists a constant $d > 0$ such that $h(t) \leq \frac{d}{t^{\alpha}}$ for all $t \in (0,\epsilon)$ where $\alpha \in (0,1)$ and $\epsilon \approx 0$;
(H3) h is decreasing.

We consider classes of C^1 reaction terms $f : [0,\infty) \to \mathbb{R}$ satisfying the following:

(F1) $f(0) < 0$ and $\lim_{s\to\infty} \frac{f(s)}{s} = 0$;
(F2) f is increasing and $\lim_{s\to\infty} f(s) = \infty$;
(F3) f is concave on $[0,\infty)$.

We establish:

Theorem 1.1. *Assume (H1)–(H3) and (F1)–(F3). Then (1.3) has a unique positive solution for all λ sufficiently large.*