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Extensions and traces of functions of bounded variation on metric spaces

Panu Lahti

Aalto University, School of Science, Department of Mathematics and Systems Analysis, P.O. Box 11100, FI-00076 Aalto, Finland

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ABSTRACT

In the setting of a metric space equipped with a doubling measure and supporting a Poincaré inequality, and based on results by Björn and Shanmugalingam (2007) [7], we show that functions of bounded variation can be extended from any bounded uniform domain to the whole space. Closely related to extensions is the concept of boundary traces, which have previously been studied by Hakkarainen et al. (2014) [17]. On spaces that satisfy a suitable locality condition for sets of finite perimeter, we establish some basic results for the traces of functions of bounded variation. Our analysis of traces also produces novel pointwise results on the behavior of functions of bounded variation in their jump sets.

1. Introduction

A classical Euclidean result on the extension of Sobolev functions and functions of bounded variation, abbreviated as BV functions, is that any bounded domain with a Lipschitz boundary is an extension domain, see e.g. [2, Proposition 3.21]. For Sobolev functions, this result was generalized to so-called (ε, δ)-domains by Jones [19]. Moreover, a characterization of BV extension domains in $\mathbb{R}^2$ is given in [21]. On metric spaces, extension results for various classes of functions, such as Hajłasz–Sobolev functions and Hölder continuous functions, have been derived in e.g. [16] and [6]. One result on the extension of BV functions on metric spaces is given by Baldi and Montefalcone [4] who show, in essence, that if sets of finite perimeter can be extended from a domain, then so can general BV functions. However, a simple geometric condition ensuring the extendability of BV functions appears to be missing.

Björn and Shanmugalingam show in [7] that every *uniform domain* Ω is an extension domain for Newton–Sobolev functions $N^{1,p}(\Omega)$, with $p \geq 1$. On the other hand, BV functions on metric spaces are defined through relaxation by means of Newton–Sobolev functions, see [1] and [22]. Thus the extension result of [7]

E-mail address: panu.lahti@aalto.fi.

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can be applied to the class BV in a fairly straightforward manner, as presented in this note. It should be mentioned that bounded domains with a Lipschitz boundary are uniform domains in Euclidean spaces and also in Carnot groups, according to [12].

On the other hand, the concepts of extensions and *boundary traces* are closely related. Classical treatments of boundary traces of BV functions can be found in e.g. [2, Chapter 3] and [14, Chapter 2], whereas similar results in Carnot–Carathéodory spaces are given in [24,9,8]. A standard assumption in these results is again a Lipschitz boundary. On the other hand, in the metric setting, results on boundary traces seem to be largely absent, with the exception of [17], where boundary traces of BV functions are defined on the boundaries of certain BV extension domains.

In this note, we present a different approach to traces, where more is assumed of the space but less of the domain. More precisely, we assume a certain locality condition that essentially states that any two sets of finite perimeter "look the same" near almost every point in which their measure theoretic boundaries intersect. Then we can prove the existence of *interior traces* of BV functions on the measure theoretic boundary of any set of finite perimeter, and also prove the existence of boundary traces on the measure theoretic boundary of any BV extension domain.

In [20], pointwise properties of BV functions on metric spaces were studied, and in particular a Lebesgue point theorem for BV functions outside their jump sets was given. Since the super-level sets of a BV function are sets of finite perimeter, we are able to apply our analysis of traces to prove novel pointwise results on the behavior of a BV function in its jump set, extending classical results to metric spaces and strengthening results found in [20].

2. Preliminaries

In this section we introduce the necessary definitions and assumptions.

In this paper, (X, d, μ) is a complete metric space equipped with a Borel regular outer measure μ. The measure is assumed to be doubling, meaning that there exists a constant $c_d > 0$ such that

$$0 < \mu\big(B(x, 2r)\big) \le c_d \mu\big(B(x, r)\big) < \infty$$

for every ball $B = B(x, r)$ with center $x \in X$ and radius $r > 0$. By iterating the doubling condition, we get

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \ge C\left(\frac{r}{R}\right)^Q \tag{2.1}$$

for every $0 < r \le R$ and $y \in B(x, R)$, and some $Q > 1$ and $C > 0$ that only depend on c_d. In general, C will denote a positive constant whose value is not necessarily the same at each occurrence.

We recall that a complete metric space endowed with a doubling measure is proper, that is, closed and bounded sets are compact. Since X is proper, for any open set $\Omega \subset X$ we define e.g. $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ as the space of functions that are Lipschitz in every $\Omega' \Subset \Omega$. Here $\Omega' \Subset \Omega$ means that Ω' is open and that $\overline{\Omega'}$ is a compact subset of Ω.

For any set $A \subset X$ and $0 < R < \infty$, the restricted spherical Hausdorff content of codimension 1 is defined as

$$\mathcal{H}_R(A) := \inf\left\{\sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i \le R\right\}.$$

The Hausdorff measure of codimension 1 of a set $A \subset X$ is

$$\mathcal{H}(A) := \lim_{R \to 0} \mathcal{H}_R(A).$$