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Pitt's inequality and the uncertainty principle associated with the quaternion Fourier transform

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ABSTRACT

The quaternion Fourier transform – a generalized form of the classical Fourier transform – has been shown to be a powerful analyzing tool in image and signal processing. This paper investigates Pitt's inequality and uncertainty principle associated with the two-sided quaternion Fourier transform. It is shown that by applying the symmetric form $f = f_1 + \mathbf{i}f_2 + f_3\mathbf{j} + \mathbf{i}f_4\mathbf{j}$ of quaternion from Hitzer and the novel module or L^p-norm of the quaternion Fourier transform $\widehat{f}$, then any nonzero quaternion signal and its quaternion Fourier transform cannot both be highly concentrated. Two part results are provided, one part is Heisenberg–Weyl's uncertainty principle associated with the quaternion Fourier transform. It is formulated by using logarithmic estimates which may be obtained from a sharp of Pitt's inequality; the other part is the uncertainty principle of Donoho and Stark associated with the quaternion Fourier transform.

1. Introduction

The uncertainty principle of harmonic analysis states that a non-trivial function and its Fourier transform (FT) cannot both be sharply localized. The uncertainty principle plays an important role in signal processing [8–10,15,16,24,27,29,31,34], and physics [1,7,19–21,25,26,32,35,36]. In quantum mechanics an uncertainty principle asserts that one cannot make certain of the position and velocity of an electron (or any particle) at the same time. That is, increasing the knowledge of the position decreases the knowledge of the velocity or momentum of an electron. In quaternionic analysis some papers combine the uncertainty relations and the quaternionic Fourier transform (QFT) [2,18,22,28,30]. Recently, Heisenberg's uncertainty relations were extended to the quaternion linear canonical transform [22] – a generalized form of the QFT.

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The QFT plays a vital role in the representation of (hypercomplex) signals. It transforms a real (or quaternionic) $\mathbb{R}^d$ signal into a quaternion-valued frequency domain signal. The four components of the QFT separate four cases of symmetry into real signals instead of only two as in the complex FT. Due to the noncommutative property of multiplication of quaternion, there are three different types of QFT: left-sided, right-sided and two-sided QFT. Hitzer [17] introduced these different types of QFT and investigated their important properties. In [33] the authors used the QFT to proceed with color image analysis. The paper [3] implemented the QFT to design a color image digital watermarking scheme. The authors in [4] applied the QFT to image pre-processing and neural computing techniques for speech recognition.

There are different types of uncertainty relations, including entropy-based uncertainty relations, Heisenberg's uncertainty for time spread and frequency spread, the uncertainty relations for time-frequency distribution and so on. In this paper, we will mainly focus on the new inequalities for the two-sided QFT, including Pitt's inequality [5] in QFT domains, the logarithmic uncertainty relations in QFT domains, Heisenberg–Weyl's uncertainty relations in QFT domains, the uncertainty relations of Donoho and Stark [11] in QFT domains etc. To the best of our knowledge, the study of Pitt's inequality and its Heisenberg–Weyl's uncertainty relations associated with two-sided QFT has not been carried out yet. The results in this paper are new in the literature. The main motivation of the present study is to develop further technical applications in the theory of partial differential equations [12]. We would like to apply these ideas to the existence and smoothness of solutions of PDE, construction of explicit fundamental solutions, and eigenvalues of Schrödinger operators in the Hamiltonian quaternionic algebra. Further investigations and extensions of this topic will be reported in a forthcoming paper.

The paper is organized as follows. Section 2 gives a brief introduction to some general definitions and basic properties of quaternion analysis. The QFT of $\mathbb{R}^d$ quaternionic signal is introduced and studied in Section 3. Some important properties such as Plancherel's and inversion theorems are obtained. The classical Pitt's inequality and logarithmic uncertainty principle are generalized in the quaternion Fourier domains in Section 4. In Section 5 the uncertainty principle of Donoho and Stark associated with two-sided quaternion Fourier transform is provided. Finally conclusions are summarized in Section 6.

2. Preliminaries

For convenience of further discussions, we briefly review some notions and terminology on quaternion. We write

$$x \cdot u = \sum_{i=1}^{d} x_i u_i$$

for the inner product on $\mathbb{R}^d$, and abbreviate $x^2 = x \cdot x$. The Euclidean norm is defined by $|x| := \sqrt{x \cdot x}$.

Let $\mathbb{H}$ denote the quaternion algebra over $\mathbb{R}$, which is an associative non-commutative four-dimensional algebra

$$\mathbb{H} := \{q = q_1 + \mathbf{i}q_2 + \mathbf{j}q_3 + \mathbf{k}q_4 \mid q_1, q_2, q_3, q_4 \in \mathbb{R}\}, \tag{1}$$

where the elements $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ obey Hamilton's multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1, \qquad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \qquad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \qquad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{2}$$

The conjugate of a quaternion q is defined by

$$\bar{q} := q_1 - \mathbf{i}q_2 - \mathbf{j}q_3 - \mathbf{k}q_4, \tag{3}$$