J. Math. Anal. Appl. 423 (2015) 734–745

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Limit cycles of Abel equations of the first kind ☆

A. Álvarez, J.L. Bravo *, M. Fernández

Departamento de Matemáticas, Universidad de Extremadura, Avda. de Elvas s/n, 06006 Badajoz, Spain

ARTICLE INFO

Article history:
Received 20 May 2014
Available online 13 October 2014
Submitted by T. Witelski

Keywords:
Periodic solutions
Limit cycles
Abel equation

ABSTRACT

Consider the scalar differential equation $x' = \sum_{i=0}^{m} a_i(t)x^{n_i}$, where $a_i(t)$ are T-periodic analytic functions, and $1 \leq n_i \leq n$. For any polynomial $Q(x) = x^{n_0} - \sum_{i=1}^{m} \alpha_i x^{n_i}$, the equation can be written as $x' = a_0 Q(x) + R(t, x)$. Let W be the Wronskian of Q and R with respect to x, and $\tilde{Q}$, $\tilde{W}$ the previous polynomials after removing multiplicity of roots and solutions of the differential equation. We prove that if the vector field defined by the differential equation is "transversal" at every point of $\tilde{Q}(x) = 0$ or $\tilde{W}(t, x) = 0$ then the number of limit cycles (isolated periodic solutions in the set of periodic solutions) of the differential equation is at most $3n - 1$.

1. Introduction and main results

Consider the differential equation

$$\frac{dx}{dt} = x' = \sum_{i=0}^{m} a_i(t)x^{n_i} := P(t, x) \tag{1}$$

where $1 \leq n_i \leq n$, and the $a_i(t)$ are T-periodic analytic functions.

The problem of determining the number of limit cycles of (1) goes back to S. Smale and C. Pugh, and is motivated by Hilbert's 16th problem since, for some important families of polynomial planar systems, the problem of determining the limit cycles around a critical point is equivalent to determining the limit cycles of (1) with $a_i(t)$ being trigonometric polynomials (see [8,11,12,16]).

A second motivation for the study of (1) is its application in modelling real-world phenomena (see [4,9,13,19] and references therein).

☆ The authors are partially supported by MCYT/FEDER grant number MTM 2011-22751. The first author is also partially supported by Junta de Extremadura/FEDER grant number GR10007, and the last two authors by Junta de Extremadura/FEDER grant number GR10060.

* Corresponding author.
E-mail addresses: aalarma@unex.es (A. Álvarez), trinidad@unex.es (J.L. Bravo), ghierro@unex.es (M. Fernández).

http://dx.doi.org/10.1016/j.jmaa.2014.10.019

When $n = 3$, A. Lins Neto [16] proved that for any N there exist trigonometric polynomials $a_i(t)$ such that (1) has at least N limit cycles. This implies that, to bound the number of limit cycles, some additional conditions must be imposed. Most of the partial results that have been obtained impose the condition that some of the coefficients have definite sign. To cite just some of the most relevant, V.A. Pliss [17] proved that when $n = 3$ and the leading coefficient of $P(t, x)$ as a polynomial in x has definite sign then (1) has at most three limit cycles. Yu. Ilyashenko [15] obtained a bound of the solutions of (1) when the leading coefficient is 1, in terms of the ∞-norm of the rest of the coefficients. A. Gasull and J. Llibre [10] proved that when $n = 3$ and the coefficient of x^2 has definite sign then (1) has at most three limit cycles.

When the coefficients of (1) have no definite sign, there are two families of results. The first imposes symmetries (and signs) on some of the coefficients (see for instance [6,7]), while the second requires that some linear combination of the coefficients has definite sign and imposes some restrictions on the possible monomials of (1) (see [1,5,14]; for more details, we refer the reader to the last subsection of the present paper).

In this communication, we generalize this second family of results in such a way that they can be applied to general families of Abel equations (previous results impose that the maximum number of different degrees in x is three). In order to precise the results, let us fix $\alpha_i \in \mathbb{R}$, $0 \le i \le m$. We may rewrite (1) as

$$x' = a_0(t)Q(x) + R(t, x), \tag{2}$$

where

$$Q(x) := x^{n_0} - \sum_{i=1}^{m} \alpha_i x^{n_i}, \qquad R(t, x) := \sum_{i=1}^{m} b_i(t)x^{n_i}, \quad b_i(t) = \alpha_i a_0(t) + a_i(t).$$

Let $u(t, x)$ denote the solution of (1) determined by $u(0, x) = x$. Note that $u(t, x)$ is periodic if and only if $u(T, x) = x$. Therefore, isolated zeroes of the displacement function, $\Delta(x) := u(T, x) - x$, are the limit cycles of (1).

In Proposition 2.1, we shall prove that if $u(t, x)$ is a limit cycle of (2) such that $Q(u(t)) \neq 0$ in $[0, T]$ then

$$u_x(T, x) = \exp\left(\int_0^T \frac{W(t, u(t, x))}{Q(u(t, x))}\, dt\right),$$

where W is the Wronskian of Q and R as functions in x, i.e.,

$$W(t, x) = Q(x)R_x(t, x) - R(t, x)Q'(x).$$

(Throughout this paper, we shall use subindices in a function to denote the corresponding partial derivatives.)

In particular, we shall prove that this implies that, if V is a connected component of

$$U = \{(t, x) \in \mathbb{R}^2 : W(t, x)Q(x) \neq 0\},$$

then there is at most one limit cycle with trajectory included in V. Under additional conditions which imply that limit cycles are either contained in U or disjoint with U, Theorem 1.1 will give the upper bound $3n - 1$ (the maximum number of connected components of U) for the number of limit cycles of (1).

What are the precise conditions ensuring that limit cycles are either contained in or disjoint with U? Since $a_i(t)$ and $b_i(t)$ are periodic and analytic, (1) is defined on the cylinder $\mathbb{S}^1 \times \mathbb{R}$, and all the functions considered are polynomials in x with coefficients in the domain of analytic functions $\mathcal{C}^w(\mathbb{S}^1)$. In particular, they belong to the principal ideal domain of polynomials $M(\mathbb{S}^1)[x]$, where $M(\mathbb{S}^1)$ is the field of fractions